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On disjoint matchings in cubic graphs

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ABSTRACT

For $i = 2, 3$ and a cubic graph G let $\nu_i(G)$ denote the maximum number of edges that can be covered by i matchings. We show that $\nu_2(G) \geq \frac{4}{5}|V(G)|$ and $\nu_3(G) \geq \frac{7}{6}|V(G)|$. Moreover, it turns out that $\nu_2(G) \leq \frac{|V(G)|+2\nu_3(G)}{4}$.

1. Introduction

In this paper graphs are assumed to be finite, undirected and without loops, though they may contain multiple edges. We will also consider pseudo-graphs, which, in contrast with graphs, may contain loops. Thus graphs are pseudo-graphs. We accept the convention that a loop contributes to the degree of a vertex by two.

The set of vertices and edges of a pseudo-graph G will be denoted by $V(G)$ and $E(G)$, respectively. We also define: $n = |V(G)|$ and $m = |E(G)|$. We will also use the following scheme for notations: if G is a pseudo-graph and f is a graph-theoretic parameter, we will write just f instead of $f(G)$. So, for example, if we would like to deal with the edge-set of a pseudo-graph $G_i^{(0)*}$, we will write $E_i^{(0)*}$ instead of $E(G_i^{(0)*})$; moreover we will write $m_i^{(0)*}$ for the number of edges in this graph.

A connected 2-regular graph with at least two vertices will be called a *cycle*. Thus, a loop is not considered to be a cycle in a pseudo-graph. Note that our notion of cycle differs from the cycles that people working on nowhere-zero flows and cycle double covers are used to deal with.

The length of a path or a cycle is the number of edges lying on it. The path or cycle is even (odd) if its length is even (odd). Thus, an isolated vertex is a path of length zero, and it is an even path.

For a graph G let $\Delta = \Delta(G)$ and $\delta = \delta(G)$ denote the maximum and minimum degrees of vertices in G, respectively. Let $\chi' = \chi'(G)$ denote the chromatic class of the graph G.

The classical theorem of Shannon states:

Theorem 1 (*Shannon [18]*). *For every graph G*

$$\Delta \leq \chi' \leq \left[\frac{3\Delta}{2}\right]. \tag{1}$$

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In 1965, Vizing proved:

Theorem 2 (*Vizing [21]*). *$\Delta \leq \chi' \leq \Delta + \mu$, where μ denotes the maximum multiplicity of an edge in G.*

Note that Shannon's theorem implies that if we consider a cubic graph G, then $3 \leq \chi' \leq 4$, thus χ' can take only two values. In 1981 Holyer proved that the problem of deciding whether $\chi' = 3$ or not for cubic graphs G is NP-complete [9], thus the calculation of χ' is already hard for cubic graphs.

For a graph G and a positive integer k define

$$B_k \equiv \{(H_1, \ldots, H_k) : H_1, \ldots, H_k \text{ are pairwise edge-disjoint matchings of } G\},$$

and let

$$\nu_k \equiv \max\{|H_1| + \cdots + |H_k| : (H_1, \ldots, H_k) \in B_k\}.$$

Define:

$$\alpha_k \equiv \max\{|H_1|, \ldots, |H_k| : (H_1, \ldots, H_k) \in B_k \text{ and } |H_1| + \cdots + |H_k| = \nu_k\}.$$

If ν denotes the cardinality of the largest matching of G, then it is clear that $\alpha_k \leq \nu$ for all G and k. Moreover, $\nu_k = |E| = m$ for all $k \geq \chi'$. Let us also note that ν_1 and α_1 coincide with ν.

In contrast with the theory of 2-matchings, where every graph G admits a maximum 2-matching that includes a maximum matching [11], there are graphs that do not have a "maximum" pair of disjoint matchings (a pair $(H, H') \in B_2$ with $|H| + |H'| = \nu_2$) that includes a maximum matching.

The following is the best result that can be stated about the ratio ν/α_2 for any graph G (see [15]):

$$1 \leq \nu/\alpha_2 \leq 5/4. \tag{2}$$

Very deep characterization of graphs G satisfying $\nu/\alpha_2 = 5/4$ is given in [20].

Let us also note that by Mkrtchyan's result [12], reformulated as in [6], if G is a matching covered tree, then $\alpha_2 = \nu$. Note that a graph is said to be matching covered (see [13]), if its every edge belongs to a maximum matching (not necessarily a perfect matching as it is usually defined, see e.g. [11]).

The basic problem that we are interested is the following: what is the proportion of edges of an r-regular graph (particularly, cubic graph), that we can cover by its k matchings? The formulation of our problem stems from the recent paper [10], where the authors investigate the proportion of edges of a bridgeless cubic graph that can be covered by k of its perfect matchings.

The aim of the present paper is the investigation of the ratios $\nu_k/|E|$ (or equivalently, $\nu_k/|V|$) in the class of cubic graphs for $k = 2, 3$. Note that for cubic graphs G Shannon's theorem implies that $\nu_k = |E|$, $k \geq 4$.

The case $k = 1$ has attracted much attention in the literature. See [8] for the investigation of the ratio in the class of simple cubic graphs, and [3,7,17,16,22] for the general case. Let us also note that the relation between ν_1 and $|V|$ has also been investigated in the regular graphs of high girth [4].

The same is true for the case $k = 2, 3$. Albertson and Haas investigate these ratios in the class of simple cubic graphs (i.e. graphs without multiple edges) in [1,2], and Steffen investigates the general case in [19].

2. Some auxiliary results

If G is a pseudo-graph, and $e = (u, v)$ is an edge of G, then k-subdivision of the edge e results a new pseudo-graph G' which is obtained from G by replacing the edge e with a path P_{k+1} of length $k + 1$, for which $V(P_{k+1}) \cap V = \{u, v\}$. Usually, we will say that G' is obtained from G by k-subdividing the edge e.

If Q is a path or cycle of a pseudo-graph G, and the pseudo-graph G' is obtained from G by k-subdividing the edge e, then sometimes we will speak about the path or cycle Q' corresponding to Q, which roughly, can be defined as Q, if e does not lie on Q, and the path or cycle obtained from Q by replacing its edge e with the path P_{k+1}, if e lies on Q.

Our interest towards subdivisions is motivated by the following

Proposition 1. *Let G be a connected graph with $2 \leq \delta \leq \Delta = 3$. Then, there exists a connected cubic pseudo-graph G_0 and a mapping $k : E_0 \to Z^+$, such that G is obtained from G_0 by $k(e)$-subdividing each edge $e \in E_0$, where Z^+ is the set of non-negative integers.*

Proof. The existence of such a cubic pseudo-graph G_0 can be verified, for example, as follows; as the vertex-set of G_0, we take the set of vertices of G having degree three, and connect two vertices u, v of G_0 by an edge $e = (u, v)$, if these vertices are connected by a path P of length k, $k \geq 1$ in G, whose end-vertices are u and v, and whose internal vertices are of degree two. We also define $k(e) = k - 1$. Finally, if a vertex w of G_0 lies on a cycle C of length l, $l \geq 1$ in G, whose all vertices, except w, are of degree two, then in G_0 we add a loop f incident to w, and define $k(f) = l - 1$. Now, it is not hard to verify, that G_0 is a cubic pseudo-graph, and if we $k(d)$-subdivide each edge d of G_0, then the resulting graph is isomorphic to G. $\square$

Let G_0 be a cubic pseudo-graph, and let e be a loop of G_0. Let f be the edge of G_0 adjacent to e (note that f is not a loop). Let u_0 be the vertex of G_0 that is incident to f and e, and let $f = (u_0, v_0)$. Assume that v_0 is not incident to a loop of G_0, and let h and h' be the other ($\neq f$) edges of G_0 incident to v_0, and assume u and v be the endpoints of h and h', that are not incident

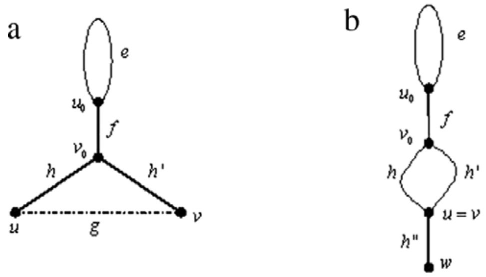

Fig. 1. Cutting a loop *e*.

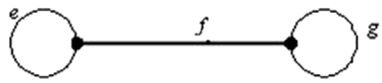

Fig. 2. The trivial case.

to f, respectively. Consider the cubic pseudo-graph G_0' obtained from G_0 as follows ((a) of Fig. 1):

$$G_0' = (G_0 \setminus \{u_0, v_0\}) \cup \{g\}, \quad \text{where } g = (u, v).$$

Note that u and v may coincide. In this case g is a loop of G_0'. We will say that G_0' is obtained from G_0 by cutting the loop e.

People dealing especially with bridgeless cubic graphs would have already recognized Fleischner's splitting off operation. Completely realizing this, we would like to keep the name "cutting the loops", in order to keep the basic idea, that has helped us to come to its definition!

Remark 1. If G_0 is a connected cubic pseudo-graph, then the successive cut of loops of G_0 in any order of loops leads either to a connected graph (that is, connected pseudo-graph without loops), or to the cubic pseudo-graph shown on the Fig. 2. Sometimes, we will prefer to restate this property in terms of applicability of the operation of cutting the loop. More specifically, if G_0 is a connected cubic pseudo-graph, for which the operation of cutting the loop is not applicable, then either G_0 does not contain a loop or it is the mentioned trivial graph.

Before we move on, we would like to state some properties of the operation of cutting the loops.

Proposition 2. *If G_0 is connected, then G_0' is connected, too.*

Proposition 3. *If a connected cubic pseudo-graph G_0 contains a cycle, and a cubic pseudo-graph G_0' obtained from G_0 by cutting a loop e of G_0 does not, then e is adjacent to an edge f, which, in its turn, is adjacent to two edges h and h', that form the only cycle of G_0 with length two ((b) of Fig. 1).*

The following will be used frequently:

Proposition 4. *Let be a, b, c, d be positive numbers with $\frac{a}{b} \geq \alpha$, $\frac{c}{d} \geq \alpha$. Then:*

$$\frac{a+c}{b+d} \geq \alpha. \tag{3}$$

Proposition 5. *Suppose that $x_1 \leq y_1, x_2 \geq y_2, \ldots, x_n \geq y_n, x_1 + \cdots + x_n = y_1 + \cdots + y_n$ and $\min_{1 \leq i \leq n}\{\alpha_i\} = \alpha_1 > 0$. Then:*

$$\alpha_1 x_1 + \cdots + \alpha_n x_n \geq \alpha_1 y_1 + \cdots + \alpha_n y_n. \tag{4}$$

Proof. Note that

$$\alpha_2(x_2 - y_2) \geq \alpha_1(x_2 - y_2),$$
$$\vdots$$
$$\alpha_n(x_n - y_n) \geq \alpha_1(x_n - y_n),$$

thus

$$\alpha_2(x_2 - y_2) + \cdots + \alpha_n(x_n - y_n) \geq \alpha_1(x_2 - y_2) + \cdots + \alpha_1(x_n - y_n) = \alpha_1(y_1 - x_1) \tag{5}$$

or

$$\alpha_1 x_1 + \cdots + \alpha_n x_n \geq \alpha_1 y_1 + \cdots + \alpha_n y_n. \quad \square \tag{6}$$

Theorem 3 (*Gallai [11]*). *Let G be a connected graph with $\nu(G-u)=\nu$ for any $u\in V$. Then G is factor-critical, and particularly:*

$$n=2\nu+1.$$

Terms and concepts that we do not define can be found in [5,11,23].

3. Maximum matchings and unsaturated vertices

In this section we prove a lemma, which states that, under some conditions, one can always pick up a maximum matching of a graph, such that the unsaturated vertices with respect to this matching "are not placed very close".

Before we present our result, we would like to deduce a lower bound for ν in the class of regular graphs using the Theorem 3 of Gallai.

Observe that Shannon's theorem implies that $\chi'\leq 4$ for every cubic graph G, thus $m\leq 4\nu=4\nu_1$. Now, it turns out, that there are no cubic graphs G, for which $m=4\nu_1$, thus $\nu_1>\frac{m}{4}$. Next we prove a generalization of this statement, that originally appeared in [14] as a problem:

Lemma 1. (a) *No $(2k+1)$-regular graph G contains $2k+2$ pairwise edge-disjoint maximum matchings;*
(b) *If G is a connected simple r-regular graph with $r+1$ pairwise edge-disjoint maximum matchings, then r is even and G is the complete graph.*

Proof. (a) Assume G to contain $2k+2$ pairwise edge-disjoint maximum matchings $F_1,\ldots,F_{2k+2}$. Note that we may assume G to be connected. Clearly, for every $v\in V$ there is $F_v\in\{F_1,\ldots,F_{2k+2}\}$ such that F_v does not saturate the vertex v. By a Theorem 3 of Gallai, it follows that $n=2\nu+1$, that is, n is odd, which is impossible.

(b) Assume G to contain $r+1$ pairwise edge-disjoint maximum matchings $F_1,\ldots,F_{r+1}$. (a) implies that $n=2\nu+1$ and r is even. Since, by Vizing's theorem $\chi'\leq r+1$, we have

$$(r+1)\nu=|F_1|+\cdots+|F_{r+1}|\leq m\leq\chi'\nu\leq(r+1)\nu,$$

thus

$$(r+1)\frac{n-1}{2}=(r+1)\nu=m=r\frac{n}{2},$$

or

$$r=n-1,$$

hence G is the complete graph. $\square$

Remark 2. As the example of the "fat triangle" shows, the complete graph with odd number of vertices is not the only graph, that prevents us to generalize (a) to even regular graphs.

Next we prove the main result of the section, which is interesting not only on its own, but also will help us to derive better bounds in the Theorem 4.

Lemma 2. *Every graph G, with $2\leq\delta\leq\Delta\leq 3$, contains a maximum matching, such that the unsaturated vertices (with respect to this maximum matching) do not share a neighbour.*

Proof. Let F be a maximum matching of G, for which there are minimum number of pairs of unsaturated vertices, which have a common neighbour. The lemma will be proved, if we show that this number is zero.

Suppose that there are vertices u and w of G which are not saturated (by F) and have a common neighbour q. Clearly, q is saturated by an edge $e_q\in F$. Consider the edge $e=(u,q)$. Note that it lies in a maximum matching of G (an example of such a maximum matching is $(F\backslash\{e_q\})\cup\{e\}$). Moreover, for every maximum matching F_e of G with $e\in F_e$, the alternating component P_e of $F\Delta F_e$ which contains the edge e, is a path of even length. Now, choose a maximum matching F' of G containing the edge e for which the length of P_e is maximum.

Let v be the other ($\neq u$) end-vertex of the path P_e. Note that since P_e is even, there is a vertex p of P_e such that $(p,v)\in F$.

Claim 1. *The neighbours of v lie on P_e and are different from u and q.*

Proof. First of all let us show that the neighbours of v lie on P_e. On the opposite assumption, consider a vertex v' which is adjacent to v and which does not lie on P_e. Clearly $(v,v')\notin F\cup F'$. As F' is a maximum matching, there is an edge $f\in F'$ incident to v'. Define:

$$F''=(F'\setminus\{f\})\cup\{(v,v')\}.$$

Note that F'' is a maximum matching of G with $e\in F''$ for which the length of the alternating component of $F\Delta F''$, which contains the edge e, exceeds the length of P_e contradicting the choice of F'. Thus the neighbours of v lie on P_e. Let us show that they are different from u and q. If there is an edge e_1 connecting the vertices u and v, then define:

$$F'''=(F\setminus E(P_e))\cup([F'\cap E(P_e)]\setminus\{e\})\cup\{e_1,(q,w)\}.$$

Clearly, F''' is a matching of G for which $\left|F'''\right| > |F|$, which is impossible. Thus, there are no edges connecting u and v. As q is adjacent to u and w, v can be adjacent to q if and only if $p = q$, that is, if the length of P_e is two. But this is impossible, too, since $d_G(v) \geq 2$, hence there should be an edge connecting u and v. The proof of Claim 1 is completed. □

Corollary 1. *The length of P_e is at least four.*

To complete the proof of the lemma we need to consider two cases:

Case 1: $(p, w) \notin E$.

Consider a maximum matching F_0 of G which is obtained from F by shifting the edges of F on P_e, that is,

$$F_0 = (F \setminus E(P_e)) \cup (F' \cap E(P_e)).$$

Note that F_0 saturates all vertices of P_e except v. Consider a vertex v_0 which is a neighbour of v. Due to Claim 1, v_0 is a vertex of P_e, which is different from u and q. Note that the neighbours of v_0 are the vertex v and one or two other vertices of P_e which are saturated by F_0. Thus there is no unsaturated vertex of G, which has a common neighbour with v. This implies that the number of pairs of vertices of G which are not saturated by F_0 and have a common neighbour is less than the corresponding number for F, which contradicts the choice of F.

Case 2: $(p, w) \in E$.

Consider a maximum matching F_1 of G, defined as:

$$F_1 = (F \setminus \{(p, v)\}) \cup \{(p, w)\}.$$

Note that F_1 saturates w and does not saturate v. Consider a vertex v_1 which is a neighbour of v. Due to Claim 1, v_1 is a vertex of P_e, which is different from u and q. Note that the neighbours of v_1 are the vertex v and two other vertices of P_e which are saturated by F_1. Thus there is no unsaturated vertex of G, which has a common neighbour with v. This implies that the number of pairs of vertices of G which are not saturated by F_1 and have a common neighbour is less than the corresponding number for F, which contradicts the choice of F. The proof of Lemma 2 is completed. □

It would be interesting to generalize the statement of Lemma 2 to almost regular graphs. In other words, we would like to suggest the following

Conjecture 1. *Let G be graph with $\Delta - \delta \leq 1$. Then G contains a maximum matching such that the unsaturated vertices (with respect to this maximum matching) do not share a neighbour.*

We would like to note that we do not even know, whether the conjecture holds for r-regular graphs with $r \geq 4$.

4. The system of cycles and paths

In this section we prove two lemmas. For graphs that belong to a very peculiar family, the first of them allows us to find a system of cycles and paths that satisfy some explicitly stated properties. The second lemma helps in finding a system with the same properties in graphs that are subdivisions of the graphs from the mentioned peculiar class. Moreover, due to the second lemma, it turns out that if there is a system of the original graph that includes a maximum matching, then there is a system of the subdivided graph preserving this property!

Lemma 3. *Let G be a graph with $\delta \geq 2$. Suppose that every edge of G connects a vertex of degree two to one with degree at least three. Then*

(1) *there exists a vertex-disjoint system of even paths $P_1, \ldots, P_r$ and cycles $C_1, \ldots, C_l$ of G such that*
  (1.1) $r = \frac{1}{2}\sum_{v, d(v)\geq 3}(d(v) - 2)$;
  (1.2) *all vertices of G lie on these paths or cycles;*
  (1.3) *the end-vertices of the paths $P_1, \ldots, P_r$ are of degree two and these end-vertices are adjacent to vertices of degree at least three;*

(2) *for every maximum matching F of G, every pair of edge-disjoint matchings (H, H') with $|H| + \left|H'\right| = \nu_2$, every vertex $v \in V$ with $d(v) \geq 3$, is incident to one edge from F, one from H and one from H'.*

(3) *G contains two edge-disjoint maximum matchings;*

(4) *if $\delta = 2$, $\Delta = k \geq 3$, $d(v) \in \{2, k\}$ for every vertex $v \in V$, then*

$$\nu_1 = \frac{2}{k+2}n, \qquad \nu_2 = \frac{4}{k+2}n. \tag{7}$$

Proof. (1) Clearly, G is a bipartite graph, since the sets

$$V_2 = \{v \in V : d(v) = 2\},$$
$$V_{\geq 3} = \{v \in V : d(v) \geq 3\}$$

form a bipartition of G. We intend to construct a system of pairwise vertex-disjoint cycles and even paths of G such that the all vertices of $V_{\geq 3}$ lie on them. Of course, the cycles will be of even length since G is bipartite.

Choose a system of cycles $C_1, \ldots, C_l$ of G such that $V(C_i) \cap V(C_j) = \emptyset, 1 \le i < j \le l$ and the graph $G_0 = G \setminus (V(C_1) \cup \cdots V(C_l))$ does not contain a cycle. Clearly, G_0 is a forest, that is, a graph every component of which is a tree. Moreover, for every $v_0 \in V_0$

(a) if $d_{G_0}(v_0) \ge 3$ then $d_{G_0}(v_0) = d_G(v_0)$;
(b) if $d_{G_0}(v_0) \in \{0, 1, 2\}$ then $d_G(v_0) = 2$.

If G_0 contains no edge then add the remaining isolated vertices (paths of length zero) to the system to obtain the mentioned system of cycles and even paths of G. Otherwise, consider a non-trivial component of G_0. Let P_1 be a path of this component connecting two vertices which have degree one in G_0. Since G is bipartite, (b) implies that P_1 is of even length. Consider a graph G_1 obtained from G_0 by removing the path P_1, that is,

$$G_1 = G_0 \setminus V(P_1).$$

Note that G_1 is a forest. Moreover, it satisfies the properties (a) and (b) like G_0 does, that is, for every $v_1 \in V_1$

(a′) if $d_{G_1}(v_1) \ge 3$ then $d_{G_1}(v_1) = d_G(v_1)$;
(b′) if $d_{G_1}(v_1) \in \{0, 1, 2\}$ then $d_G(v_1) = 2$.

Clearly, by the repeated application of this procedure we will get a system of even paths $P_1, \ldots, P_{r_0}$ of G such that the graph $G \setminus (V(C_1) \cup \cdots V(C_l) \cup V(P_1) \cup \cdots V(P_{r_0})) = G_0 \setminus (V(P_1) \cup \cdots V(P_{r_0}))$ contains no edge. Now, add the remaining isolated vertices (paths of length zero) to $P_1, \ldots, P_{r_0}$ to obtained a system of even paths $P_1, \ldots, P_r$.

Note that by the construction $C_1, \ldots, C_l$ and $P_1, \ldots, P_r$ are vertex-disjoint. Moreover, the paths $P_1, \ldots, P_r$ are of even length. As G is bipartite, the cycles $C_1, \ldots, C_l$ are of even length, too.

Again, by the construction of $C_1, \ldots, C_l$ and $P_1, \ldots, P_r$ we have (1.2) and that the end-vertices of $P_1, \ldots, P_r$ are of degree two. As every edge of G connects a vertex of degree two to one with degree at least three, the system $C_1, \ldots, C_l, P_1, \ldots, P_r$ satisfies (1.3).

Let us show that (1.1) holds, too. Since the number of vertices of degree two and at least three is the same on the cycles $C_1, \ldots, C_l$, and the difference of these two numbers is one on each path from $P_1, \ldots, P_r$, then taking into account (1.2) and (1.3) we get:

$$r = |V_2| - \left|V_{\ge 3}\right| = \frac{2\,|V_2| - 2\left|V_{\ge 3}\right|}{2} = \frac{\sum\limits_{v, d(v) \ge 3} d(v) - 2\left|V_{\ge 3}\right|}{2} = \frac{1}{2} \sum_{v, d(v) \ge 3} (d(v) - 2).$$

(2) Define a pair of edge-disjoint matchings (H_0, H_0') in the following way: alternatively add the edges of $C_1, \ldots, C_l$ and $P_1, \ldots, P_r$ to H_0 and H_0'. Note that every vertex $v \in V_{\ge 3}$ is incident to one edge from H_0, one from H_0', and

$$2\nu_1 \ge \nu_2 \ge |H_0| + \left|H_0'\right| = 2\left|V_{\ge 3}\right|. \tag{8}$$

On the other hand, for every pair of edge-disjoint matchings (h, h'), every vertex $v \in V_{\ge 3}$ is incident to at most one edge from h and at most two edges from $h \cup h'$, therefore

$$\nu_1 = \max_h |h| \le \left|V_{\ge 3}\right|,$$

$$\nu_2 = \max_{h \cap h' = \emptyset} (|h| + \left|h'\right|) \le 2\left|V_{\ge 3}\right|,$$

thus (see (8))

$$\nu_1 = \left|V_{\ge 3}\right|, \qquad \nu_2 = 2\left|V_{\ge 3}\right|, \tag{9}$$

and for every maximum matching F of G, every pair of edge-disjoint matchings (H, H') with $|H| + \left|H'\right| = \nu_2$, every vertex $v \in V_{\ge 3}$ is incident to one edge from F, one from H and one from H'.

(3) directly follows from (2). (4) follows from (2) and the bipartiteness of G.

The proof of the Lemma 3 is completed. □

Lemma 4. *Let G be a connected graph satisfying the conditions:*

(a) $\delta \ge 2$;
(b) *no edge of G connects two vertices having degree at least three.*

Let G' be a graph obtained from G by a 1*-subdivision of an edge. If G contains a system of paths $P_1, \ldots, P_r$ and even cycles $C_1, \ldots, C_l$ such that*

(1) *the degrees of vertices of a cycle from $C_1, \ldots, C_l$ are two and at least three alternatively,*
(2) *all vertices of G lie on these paths or cycles;*
(3) *the end-vertices of the paths $P_1, \ldots, P_r$ are of degree two, and the vertices that are adjacent to these end-vertices and do not lie on $P_1, \ldots, P_r$ are of degree at least three;*

(4) *every edge that does not lie on* $C_1, \ldots, C_l$ *and* $P_1, \ldots, P_r$ *is incident to one vertex of degree two and one of degree at least three;*
(5) *there is a maximum matching* F *of* G *such that every edge* $e \in F$ *lies on* $C_1, \ldots, C_l$ *and* $P_1, \ldots, P_r$,

then there is a system of paths $P'_1, \ldots, P'_{r'}$ *and even cycles* $C'_1, \ldots, C'_{l'}$ *of the graph* G' *with* $r' = r$ *satisfying* (1)–(5).

Proof. Let $P_1, \ldots, P_r$ and $C_1, \ldots, C_l$ be a system of paths and even cycles satisfying (1)–(5) and let e be the edge of G whose 1-subdivision led to the graph G'. First of all we will construct a system of paths and even cycles of G' satisfying the conditions (1)–(4).

We need to consider three cases:

Case 1: e lies on a path $P \in \{P_1, \ldots, P_r\}$.

Let P' be the path of G' corresponding to P (that is, the path obtained from P by the 1-subdivision of the edge e). Consider a system of paths and even cycles of G' defined as:

$$C'_i = C_i, \quad i = 1, \ldots, l,$$
$$\{P'_1, \ldots, P'_{r'}\} = (\{P_1, \ldots, P_r\} \setminus \{P\}) \cup \{P'\}.$$

Clearly, $r' = r$. It can be easily verified that the system $P'_1, \ldots, P'_{r'}$ and $C'_1, \ldots, C'_l$ satisfies (1)–(4).

Case 2: e does not lie on either of $P_1, \ldots, P_r$ and $C_1, \ldots, C_l$.

Let w_e be the new vertex of G' and let e', e'' be the new edges of G', that is:

$$V' = V \cup \{w_e\},$$
$$E' = (E \setminus \{e\}) \cup \{e', e''\}.$$

(4) implies that e is incident to a vertex u of degree two and a vertex v of degree at least three, and suppose that $e' = (v, w_e), e'' = (w_e, u)$.

Since $d(u) = 2$ and e does not lie on $P_1, \ldots, P_r$ and $C_1, \ldots, C_l$, (2) implies that there is a path $P_u \in \{P_1, \ldots, P_r\}$ such that u is an end-vertex of P_u. Consider the path P'_u defined as:

$$P'_u = w_e, e'', P_u$$

and a system of paths and even cycles of G' defined as:

$$C'_i = C_i, \quad i = 1, \ldots, l,$$
$$\{P'_1, \ldots, P'_{r'}\} = (\{P_1, \ldots, P_r\} \setminus \{P_u\}) \cup \{P'_u\}.$$

Clearly, $r' = r$. Note that the new system satisfies (1) and (2). Let us show that it satisfies (3) and (4), too. Since $d_{G'}(w_e) = 2$, w_e is adjacent to the vertex v of degree at least three and w_e is an end-vertex of P'_u, we imply that the system $P'_1, \ldots, P'_{r'}$ and $C'_1, \ldots, C'_l$ satisfies (3).

Note that we need to verify (4) only for the edge e'. As $d_{G'}(w_e) = 2$, $d_{G'}(v) \geq 3$, we imply that the system $P'_1, \ldots, P'_{r'}$ and $C'_1, \ldots, C'_l$ satisfies (4), too.

Case 3: e lies on a cycle $C \in \{C_1, \ldots, C_l\}$.

Let w_e be the new vertex of G' and let e', e'' be the new edges of G', that is:

$$V' = V \cup \{w_e\},$$
$$E' = (E \setminus \{e\}) \cup \{e', e''\}.$$

(1) implies that the edge e is incident to a vertex u of degree two and to a vertex v of degree at least three, and suppose that $e' = (v, w_e), e'' = (w_e, u)$. Since $d_G(v) \geq 3$, (b) implies that there is a vertex $z \in V$ such that $(v, z) \in E$ and $z \notin V(C)$. Note that since $d_G(z) = 2$ and the edge (v, z) does not lie on either of $P_1, \ldots, P_r$ and $C_1, \ldots, C_l$ (2) implies that there is a path $P_z \in \{P_1, \ldots, P_r\}$ such that z is an end-vertex of P_z.

Let P be the path $C - e$ of G starting from the vertex v. Consider a path P' of G' defined as:

$$P' = P_z, (z, v), P, e'', w_e$$

and a system of paths and even cycles of G' defined as:

$$\{C'_1, \ldots, C'_{l'}\} = (\{C_1, \ldots, C_l\} \setminus \{C\})$$
$$\{P'_1, \ldots, P'_{r'}\} = (\{P_1, \ldots, P_r\} \setminus \{P_z\}) \cup \{P'\}.$$

Clearly, $r' = r$. Note that the new system satisfies (1) and (2). Let us show that it satisfies (3) and (4), too. Since $d_{G'}(w_e) = 2$, w_e is adjacent to the vertex v of degree at least three, we imply that the system $P'_1, \ldots, P'_{r'}$ and $C'_1, \ldots, C'_l$ satisfies (3).

Note that we need to verify (4) only for the edge e'. As $d_{G'}(w_e) = 2$, $d_{G'}(v) \geq 3$ we imply that the system $P'_1, \ldots, P'_{r'}$ and $C'_1, \ldots, C'_l$ satisfies (4), too.

The consideration of these three cases implies that there is a system $P'_1, \ldots, P'_{r'}$ and $C'_1, \ldots, C'_{l'}$ of paths and even cycles of G' with $r' = r$ satisfying the conditions (1)–(4). Let us show that among such systems there is at least one satisfying (5), too.

Consider all pairs $(\mathfrak{F}'_0, M'_0)$ in the graph G' where $\mathfrak{F}'_0$ is a system $P'_1, \dots, P'_{r'}$ and $C'_1, \dots, C'_{l'}$ of paths and even cycles of G' with $r' = r$ satisfying the conditions (1)–(4) and M'_0 is a maximum matching of G'. Among these choose a pair $(\mathfrak{F}', M')$ for which the number of edges of M' which lie on cycles and paths of $\mathfrak{F}'$ is maximum. We claim that all edges of M' lie on cycles and paths of $\mathfrak{F}'$.

Claim 2. *If C is a cycle from $\mathfrak{F}'$ with length $2n$ then there are exactly n edges of M' lying on C.*

Proof. Let k be the number of vertices of C which are saturated by an edge from $M' \setminus E(C)$. (1) implies that if we remove these k vertices from C we will get k paths with an odd number of vertices. Thus each of these k paths contains a vertex that is not saturated by M'. Thus the total number of edges from $M' \cap E(C)$ is at most

$$\left|M' \cap E(C)\right| \le \frac{2n-2k}{2} = n - k.$$

Consider a maximum matching M' of G' defined as:

$$M'' = (M' \setminus M'_C) \cup M''_C,$$

where M'_C is the set of edges of M' that are incident to a vertex of C, and M''_C is a 1-factor of C. Note that if $k \ge 1$ then

$$\left|M'' \cap E(C)\right| > \left|M' \cap E(C)\right|$$

and therefore for the pair $(\mathfrak{F}', M'')$ we would have that M'' contains more edges lying on cycles and paths of $\mathfrak{F}'$ then M' does, contradicting the choice of the pair $(\mathfrak{F}', M')$, thus $k = 0$, and on the cycle C from $\mathfrak{F}'$ with length $2n$ there are exactly n edges of M'. The proof of Claim 2 is completed. □

Now, we are ready to prove that all edges of M' lie on cycles and paths of $\mathfrak{F}'$. Suppose, on the contrary, that there is an edge $e' \in M'$ that does not lie on cycles and paths of $\mathfrak{F}'$. (4) implies that e' is incident to a vertex u of degree at least three and to a vertex v of degree two. (2) implies that there is a path P_v of $\mathfrak{F}'$ such that v is an end-vertex of P_v. (2) and Claim 2 imply that there is a path P_u of $\mathfrak{F}'$ such that u lies on P_u. Let w and z be the end-vertices of P_u, and let P_{wu} and P_{zu} be the subpaths of the path P_u connecting w and z to u, respectively. Consider a system $\mathfrak{F}''$ of paths and even cycles of G' defined as follows:

$$\mathfrak{F}'' = (\mathfrak{F}' \setminus \{P_u, P_v\}) \cup \{P_{zu} - u, P'\}$$

where the path P' is defined as:

$$P' = P_{wu}, (u, v), v, P_v.$$

Note that $\mathfrak{F}''$ contains exactly $r' = r$ paths. It can be easily verified that the new system $\mathfrak{F}''$ of paths and even cycles of G' satisfies (1)–(4).

Now if we consider the pair $(\mathfrak{F}'', M')$ we would have that the paths and even cycles of $\mathfrak{F}''$ include more edges of M' then the paths and even cycles of $\mathfrak{F}'$ do, contradicting the choice of the pair $(\mathfrak{F}', M')$. Thus, all edges of M' lie on cycles and paths of $\mathfrak{F}'$. The proof of the Lemma 4 is completed. □

5. The subdivision and the main parameters

The aim of this section is to prove a lemma, which claims that, under some conditions, the subdivision of an edge increases the size of the maximum 2-edge-colorable subgraph of a graph by one. This is important for us, since it enables us to control our parameters, while considering many graphs that are subdivisions of the others.

Lemma 5. *Let G be a connected graph satisfying the conditions:*

(a) $\delta \ge 2$;
(b) *G is not an even cycle;*
(c) *no edge of G connects vertices with degree at least three.*
Let G' be a graph obtained from G by a 1-subdivision of an edge. Then
(1) $\nu'_2 \ge 1 + \nu_2$;
(2) $\nu'_2 = \begin{cases} 2 + \nu_2, & \text{if } G \text{ is an odd cycle,} \\ 1 + \nu_2, & \text{otherwise.} \end{cases}$

Proof. (1) Let (H, H') be a pair of edge-disjoint matchings of G with $|H| + \left|H'\right| = \nu_2$ and let e be the edge of G whose 1-subdivision led to the graph G'. We will consider three cases:

Case 1: e lies on a $H \triangle H'$ alternating cycle C.

As G is connected and is not an even cycle, there is a vertex $v \in V(C)$ with $d_G(v) \ge 3$. Clearly, there is a vertex $u \notin V(C)$ with $d_G(u) = 2$ and $(u, v) \notin H \cup H'$. Let (u, w) be the other ($\neq (u, v)$) edge incident to u and f be an edge of C incident to v. Note that since v is incident to two edges lying on C we, without loss of generality, may assume f to be different from e. Let P_0 be a path in G whose edge-set coincides with $E(C) \setminus \{f\}$ and which starts from the vertex v. Now, assume P to be a path obtained from P_0 by adding the edge (u, v) to it, and let P' be the path of G' corresponding to P (that is, the path obtained from P by the 1-subdivision of the edge e).

Now, consider a pair of edge-disjoint matchings (H_0, H_0') of G' obtained in the following way:

- if $(u, w) \notin H$ then alternatively add the edges of P' to H_0 and H_0' beginning from H_0;
- if $(u, w) \notin H'$ then alternatively add the edges of P' to H_0 and H_0' beginning from H_0'.

Define a pair of edge-disjoint matchings (H_1, H_1') of G' as follows:

$$H_1 = (H \setminus E(C)) \cup H_0, H_1' = (H' \setminus E(C)) \cup H_0'.$$

Clearly,

$$\nu_2' \geq |H_1| + \left|H_1'\right| = 1 + |H| + \left|H'\right| = 1 + \nu_2.$$

Case 2: e lies on a $H\Delta H'$ alternating path P.

Let P' be the path of G' corresponding to P (that is, the path obtained from P by the 1-subdivision of the edge e). Consider a pair of edge-disjoint matchings (H_0, H_0') of G' obtained in the following way: alternatively add the edges of P' to H_0 and H_0'. Define:

$$H_1 = (H \setminus E(P)) \cup H_0, \qquad H_1' = (H' \setminus E(P)) \cup H_0'.$$

Clearly,

$$\nu_2' \geq |H_1| + \left|H_1'\right| = 1 + |H| + \left|H'\right| = 1 + \nu_2.$$

Case 3: $e \notin H \cup H'$.

Due to (c) there is $u \in V$ with $d_G(u) = 2$, such that e is incident to u. Let f be the other ($\neq e$) edge of G that is incident to u, and assume e' to be the edge of G' that is incident to u in G' and is different from f. Now, add the edge e' to H if $f \notin H$, and to H' if $f \notin H'$. Clearly, we constructed a pair of edge-disjoint matchings of G', which contains $1 + \nu_2$ edges, therefore

$$\nu_2' \geq 1 + \nu_2.$$

(2) Note that if G is an odd cycle then G' is an even one and $\nu_2' = 2 + \nu_2$, therefore, taking into account (1) and (b), it suffices to show that if G is not a cycle then $\nu_2' \leq 1 + \nu_2$.

Let (H, H') be a pair of edge-disjoint matchings of G' with $|H| + \left|H'\right| = \nu_2'$ and let v be the new vertex of G', that is, assume $\{v\} = V' \setminus V$. We need to consider three cases:

Case 1: $H \cup H'$ contains at most one edge incident to the vertex v.

Note that

$$\begin{aligned}\nu_2 &\geq \left|(H \cup H') \cap E\right| \geq \left|(H \cup H') \cap E(G - e)\right| \\ &\geq |H| + \left|H'\right| - 1 = \nu_2' - 1\end{aligned}$$

or

$$\nu_2' \leq 1 + \nu_2.$$

Case 2: The vertex v belongs to an alternating component of $H\Delta H'$ which is a path P_v'.

Let P_v be a path of G containing the edge e and corresponding to P_v', that is, let P_v' be obtained from P_v by the 1-subdivision of the edge e. Consider a pair of edge-disjoint matchings (H_0, H_0') of G defined as follows: alternatively add the edges of P_v to H_0 and H_0'. Define:

$$H_1 = (H \setminus E(P_v')) \cup H_0, \qquad H_1' = (H' \setminus E(P_v')) \cup H_0'.$$

Note that (H_1, H_1') is a pair of edge-disjoint matchings of G. Moreover,

$$\nu_2 \geq |H_1| + \left|H_1'\right| = |H| + \left|H'\right| - 1 = \nu_2' - 1$$

or

$$\nu_2' \leq 1 + \nu_2.$$

Case 3: The vertex v belongs to an alternating component of $H\Delta H'$ which is a cycle C_v'.

Let C_v be a cycle of G containing the edge e and corresponding to C_v', that is, let C_v' be obtained from C_v by the 1-subdivision of the edge e. As G is not a cycle, we imply that there is a vertex $w \in V(C_v')$ with $d_{G'}(w) \geq 3$. Clearly, there is a vertex $w' \in V' \setminus V(C_v')$ such that $d_{G'}(w') = 2$ and $(w, w') \in E'$. Let g be the other ($\neq (w, w')$) edge of G' incident to w'. Since w is incident to two edges lying on C_v', we imply that there is an edge $f \neq e$ such that f is incident to w. Let P_{0v} be a path of G, whose set of edges coincides with $E(C_v) \setminus \{f\}$ and starts from w. Now consider the path P_v obtained from P_{0v} by adding the edge (w, w') to it.

Consider a pair of edge-disjoint matchings (H_0, H_0') of G defined as follows:

- if $g \notin H$ then alternatively add the edges of P_v to H_0 and H_0' beginning from H_0;
- if $g \notin H'$ then alternatively add the edges of P_v to H_0 and H_0' beginning from H_0'.

Define

$$H_1 = (H \setminus E(C'_v)) \cup H_0, \qquad H'_1 = (H' \setminus E(C'_v)) \cup H'_0.$$

Note that (H_1, H'_1) is a pair of edge-disjoint matchings of G. Moreover,

$$\nu_2 \geq |H_1| + \left|H'_1\right| = |H| + \left|H'\right| - 1 = \nu'_2 - 1$$

or

$$\nu'_2 \leq 1 + \nu_2.$$

The proof of the Lemma 5 is completed. □

6. The lemma

In this section we prove a lemma that presents some lower bounds for our parameters while we consider various subdivisions of graphs. The aim of this lemma is the preparation of adequate theoretical tools for understanding the growth of our parameters depending on the numbers that the edges of graphs are subdivided. In contrast with the proofs of the statements (a), (b), (c), (h), (i), that do not include any induction, the proofs of the others significantly rely on induction. Moreover, the basic tools for proving these statements by induction are the Proposition 1 and the "loop-cut", the operation that helps us to reduce the number of loops in a pseudo-graph. To understand the dynamics of the growth of our parameters, we heavily use the Lemma 5.

Before we move on, we would like to define a class of graphs which will play a crucial role in the proof of the main result of the paper.

If G_0 is a cubic pseudo-graph such that the removal (not cut) of its loops leaves a tree (if we adopt the convention presented in [5], then we may say that the "underlying graph" of G_0 is a tree; the simplest example of such a cubic pseudo-graph is one from Fig. 2), then consider the graph G obtained from G_0 by $l(e)$-subdividing each edge e of G_0, where

$$l(e) = \begin{cases} 1, & \text{if } e \text{ is a loop,} \\ 2, & \text{otherwise.} \end{cases}$$

Define $\mathfrak{M}$ to be the class of all those graphs G that can be obtained in the mentioned way. Note that the members of the class $\mathfrak{M}$ are connected graphs.

Lemma 6. *Let G_0 be a connected cubic pseudo-graph, and consider the graph G obtained from G_0 by $k(d)$-subdividing each edge d of G_0, $k(d) \geq 1$. Suppose that, for every edge d of G_0, which is not a loop, we have: $k(d) \geq 2$. Then:*

(a) *if G_0 does not contain a loop then*
  (a1) $\nu_2 \geq \frac{7}{8}n$;
  (a2) $n \geq 4n_0$;

(b) *if G_0 contains an edge f which is adjacent to two loops e and g, then G_0 is the cubic pseudo-graph from Fig. 2 and*

$$\frac{\nu_2}{n} = \frac{k(e) + k(f) + k(g) + 1}{k(e) + k(f) + k(g) + 2};$$

(c) *if G_0 contains a loop e, then consider the cubic pseudo-graph G'_0 obtained from G_0 by cutting the loop e and the graph G' obtained from G'_0 by $k'(d')$-subdividing each edge d' of G'_0, where*

$$k'(d') = \begin{cases} k(h) + k(h') - 2 & \text{if } d' = g, \\ k(d') & \text{otherwise}. \end{cases} \tag{10}$$

Then:
  (c1) $n_0 = n'_0 + 2$;
  (c2) $n = n' + k(f) + k(e) + 4$;
  (c3) $\nu_1 \geq \nu'_1 + \left[\frac{k(f)}{2}\right] + \left[\frac{k(e)+1}{2}\right] + 1$;
  (c4) $\nu_2 \geq \nu'_2 + k(f) + k(e) + 3$;

(d) (d1) $\nu_2 \geq \frac{5}{6}n$;
  (d2) $n \geq 3n_0$;

(e) (e1) *if G_0 contains a loop e such that $k(e) \geq 2$ then $\nu_2 \geq \frac{6}{7}n$ and $n \geq \frac{7}{2}n_0$;*
  (e2) *if G_0 contains an edge f such that f is not a loop and $k(f) \geq 3$ then $\nu_2 \geq \frac{6}{7}n$ and $n \geq \frac{7}{2}n_0$;*

(f) $\nu_1 \geq \frac{3}{7}n$;

(g) *if $G \in \mathfrak{M}$ then $\nu_1 \geq \frac{6}{13}n$;*

(h) *if a cubic pseudo-graph G_0' is obtained from G_0 by cutting its loop e and if a graph G' is obtained from G_0' by $k'(d')$-subdividing each edge d' of G_0', where $k'(d')$ is defined according to (10), then if $n' \geq \frac{7}{2}n_0'$ then $n \geq \frac{7}{2}n_0$; in other words, the property $n < \frac{7}{2}n_0$ is an invariant for the operation of cutting a loop and defining k' according to (10);*

(i) *if $n < \frac{7}{2}n_0$ then $G \in \mathfrak{M}$.*

Proof. (a) For the proof of (a1) consider a graph G' obtained from G_0 by 1-subdividing each edge of G_0. Note that G' satisfies the conditions of (4) of the Lemma 3, thus (see the equality (7))

$$\nu_2' = \frac{4}{5}n' = \frac{4}{5}(n_0 + m_0) = \frac{4}{5} \cdot \frac{5}{2} \cdot n_0 = 2n_0$$

therefore due to Lemma 5 we have:

$$\frac{\nu_2}{n} = \frac{\nu_2' + \sum_{e \in E_0}(k(e) - 1)}{n' + \sum_{e \in E_0}(k(e) - 1)}. \tag{11}$$

Note that for each $e \in E_0$ $k(e) \geq 2$, hence

$$\sum_{e \in E_0}(k(e) - 1) \geq m_0 = \frac{3}{2}n_0.$$

Taking into account (11) we get:

$$\frac{\nu_2}{n} = \frac{2n_0 + \sum_{e \in E_0}(k(e) - 1)}{\frac{5}{2}n_0 + \sum_{e \in E_0}(k(e) - 1)} \geq \frac{2n_0 + \frac{3}{2}n_0}{\frac{5}{2}n_0 + \frac{3}{2}n_0} = \frac{7}{8},$$

thus

$$\nu_2 \geq \frac{7}{8}n.$$

For the proof of (a2) let us note that as G_0 does not contain a loop, for each edge f of G_0 we have $k(f) \geq 2$, thus

$$n = n_0 + \sum_{f \in E_0} k(f) \geq n_0 + 2m_0 = 4n_0.$$

(b) Note that

$$n = n_0 + k(e) + k(f) + k(g) = 2 + k(e) + k(f) + k(g).$$

Since f is not a loop, we have $k(f) \geq 2$ thus

$$\nu_2 = m - 2 = 1 + k(e) + k(f) + k(g),$$

and

$$\frac{\nu_2}{n} = \frac{k(e) + k(f) + k(g) + 1}{k(e) + k(f) + k(g) + 2}.$$

(c) The proof of (c1) follows directly from the definition of the operation of cutting loops. For the proof of (c2) note that

$$\begin{aligned} n &= n' - k'(g) + k(h) + k(h') + 1 + k(f) + 1 + k(e) \\ &= n' + k(f) + k(e) + 4 \end{aligned}$$

since $k'(g) = k(h) + k(h') - 2$ (see (10)).

For the proof of (c3) and (c4) let us introduce some additional notations. Let $C_e, P_f, P_h, P_{h'}$ be the cycle and paths of G corresponding to the edges e, f, h, h' of the cubic pseudo-graph G_0. Let K_g be the cycle or a path of G' corresponding to the edge g of the cubic pseudo-graph G_0'.

Let F' be a maximum matching of the graph G'. Define $\varepsilon = \varepsilon(F')$ as the number of vertices from $\{u, v\}$ which are saturated by an edge from $F' \cap E(K_g)$. Note that if $u \neq v$ then $0 \leq \varepsilon \leq 2$ and if $u = v$ then $0 \leq \varepsilon \leq 1$.

Consider a subset of edges of the graph G defined as:

$$F = (F' \setminus E(K_g)) \cup F_{h,h'} \cup F_f \cup F_e$$

where $F_{h,h'}$ is a maximum matching of a path $P_{h,h'}$ obtained from the paths P_h and $P_{h'}$ as follows:

$$P_{h,h'} = \begin{cases} P_h \setminus \{u, v_0\}, v_0, P_{h'} \setminus \{v_0, v\} & \text{if } \varepsilon = 0; \\ P_h \setminus \{v_0\}, v_0, P_{h'} \setminus \{v_0\} & \text{if } \varepsilon = 2; \\ P_h \setminus \{v_0\}, v_0, P_{h'} \setminus \{v_0, v\} & \text{if } \varepsilon = 1 \text{ and an edge of } F' \cap E(K_g) \text{ saturates } u; \\ P_h \setminus \{u, v_0\}, v_0, P_{h'} \setminus \{v_0\} & \text{if } \varepsilon = 1 \text{ and an edge of } F' \cap E(K_g) \text{ saturates } v; \end{cases}$$

F_f is a maximum matching of $P_f \setminus \{u_0, v_0\}$, and F_e is a maximum matching of C_e.

Note that if $u = v$ and $\varepsilon = 1$ then we define the path $P_{h,h'}$ in two ways. We would like to stress that our results do not depend on the way the path $P_{h,h'}$ is defined.

By the construction of F, F is a matching of G. Moreover,

$$\begin{aligned} \nu_1 &\geq |F| = \left|F'\right| - \left|F' \cap E(K_g)\right| + \left|F_{h,h'}\right| + \left|F_f\right| + |F_e| \\ &= \nu_1' - \left[\frac{k'(g) + \varepsilon}{2}\right] + \left[\frac{k(h) + k(h') + 1 + \varepsilon}{2}\right] + \left[\frac{k(f)}{2}\right] + \left[\frac{k(e) + 1}{2}\right] \\ &= \nu_1' - \left[\frac{k(h) + k(h') + \varepsilon}{2}\right] + 1 + \left[\frac{k(h) + k(h') + 1 + \varepsilon}{2}\right] + \left[\frac{k(f)}{2}\right] + \left[\frac{k(e) + 1}{2}\right] \\ &\geq \nu_1' + \left[\frac{k(f)}{2}\right] + \left[\frac{k(e) + 1}{2}\right] + 1 \end{aligned}$$

as

$$\left[\frac{k(h) + k(h') + 1 + \varepsilon}{2}\right] \geq \left[\frac{k(h) + k(h') + \varepsilon}{2}\right].$$

Now, let us turn to the proof of (c4). Let (H_1', H_2') be a pair of edge-disjoint matchings of G' such that $\left|H_1'\right| + \left|H_2'\right| = \nu_2'$. Define $\delta = \delta(H_1', H_2')$ as the number of vertices from $\{u, v\}$ which are saturated by an edge from $(H_1' \cup H_2') \cap E(K_g)$. Note that if $u \neq v$ then $0 \leq \delta \leq 2$ and if $u = v$ then $0 \leq \delta \leq 1$. We need to consider two cases:

Case 1: $0 \leq \delta \leq 1$;

Define a pair of edge-disjoint matchings (H_1, H_2) of G as follows:

$$\begin{aligned} H_1 &= (H_1' \setminus E(K_g)) \cup H_{1hh'} \cup H_{1fe}, \\ H_2 &= (H_2' \setminus E(K_g)) \cup H_{2hh'} \cup H_{2fe}, \end{aligned}$$

where $H_{1hh'}$, $H_{2hh'}$ are obtained from a path $P_{hh'}$ alternatively adding its edges to $H_{1hh'}$ and $H_{2hh'}$; H_{1fe}, H_{2fe} are obtained from a path P_{fe} alternatively adding its edges to H_{1fe} and H_{2fe}, and the paths $P_{hh'}$ and P_{fe} are defined as

$$P_{h,h'} = \begin{cases} P_h \setminus \{u, v_0\}, v_0, P_{h'} \setminus \{v_0, v\} & \text{if } \delta = 0; \\ P_h \setminus \{v_0\}, v_0, P_{h'} \setminus \{v_0, v\} & \text{if } \delta = 1 \text{ and an edge of } (H_1' \cup H_2') \cap E(K_g) \text{ saturates } u; \\ P_h \setminus \{u, v_0\}, v_0, P_{h'} \setminus \{v_0\} & \text{if } \delta = 1 \text{ and an edge of } (H_1' \cup H_2') \cap E(K_g) \text{ saturates } v; \end{cases}$$

$$P_{fe} = P_f \setminus \{v_0, u_0\}, u_0, C_e \setminus \{u_0\}.$$

Again, let us note that if $u = v$ and $\delta = 1$ then we define the path $P_{h,h'}$ in two ways. We would like to stress that our results do not depend on the way the path $P_{h,h'}$ is defined.

Note that

$$\begin{aligned} \nu_2 &\geq |H_1| + |H_2| = \left|(H_1' \cup H_2') \setminus E(K_g)\right| + (|H_{1hh'}| + |H_{2hh'}|) + \left(\left|H_{1fe}\right| + \left|H_{2fe}\right|\right) \\ &= \left|H_1'\right| + \left|H_2'\right| - \left|\left(H_1' \cup H_2'\right) \cap E(K_g)\right| + |E(P_{hh'})| + \left|E(P_{fe})\right| \\ &\geq \nu_2' - ((k'(g) + \delta) - 1) + ((k(h) + k(h') + \delta + 1) - 1) + ((k(f) + k(e) + 1) - 1) \\ &= \nu_2' - (k(h) + k(h') + \delta - 3) + (k(h) + k(h') + \delta) + (k(f) + k(e)) = \nu_2' + k(f) + k(e) + 3. \end{aligned}$$

Case 2: $\delta = 2$;

Define a pair of edge-disjoint matchings (H_1, H_2) of G as follows:

$$\begin{aligned} H_1 &= (H_1' \setminus E(K_g)) \cup H_{1hfe} \cup H_{1h'}, \\ H_2 &= (H_2' \setminus E(K_g)) \cup H_{2hfe} \cup H_{2h'}, \end{aligned}$$

where H_{1hfe}, H_{2hfe} are obtained from a path P_{hfe} alternatively adding its edges to H_{1hfe} and H_{2hfe}; $H_{1h'}$, $H_{2h'}$ are obtained from the path $P_{h'} \setminus \{v_0\}$ alternatively adding its edges to $H_{1h'}$ and $H_{2h'}$, and the path P_{hfe} is defined as

$$P_{hfe} = P_h \setminus \{v_0\}, v_0, P_f \setminus \{v_0, u_0\}, u_0, C_e \setminus \{u_0\}.$$

Note that

$$\begin{aligned}\nu_2 &\geq |H_1| + |H_2| = \left|(H_1' \cup H_2') \setminus E(K_g)\right| + \left(\left|H_{1hfe}\right| + \left|H_{2hfe}\right|\right) + (|H_{1h'}| + |H_{2h'}|)\\ &= \left|H_1'\right| + \left|H_2'\right| - \left|(H_1' \cup H_2') \cap E(K_g)\right| + \left|E(P_{hfe})\right| + |E(P_{h'} \setminus \{v_0\})| \geq \nu_2' - ((k'(g) + 2) - 1)\\ &\quad + (1 + k(h) + 1 + k(f) + 1 + k(e) - 1) + ((k(h') + 1) - 1)\\ &= \nu_2' - (k(h) + k(h') - 1) + (k(h) + k(f) + k(e) + 2) + k(h')\\ &= \nu_2' + k(f) + k(e) + 3.\end{aligned}$$

(d) We will give a simultaneous proof of the statements (d1) and (d2). Note that if G_0 does not contain a loop then (a1) and (a2) imply that

$$\nu_2 \geq \frac{7}{8}n > \frac{5}{6}n, \quad \text{and} \quad n \geq 4n_0 > 3n_0,$$

thus without loss of generality, we may assume that G_0 contains a loop. Our proof is by induction on n_0. Clearly, if $n_0 = 2$ then G_0 is the pseudo-graph from Fig. 2, thus (b) implies that

$$\frac{\nu_2}{n} \geq \frac{5}{6}, \quad \text{and} \quad n = 2 + k(e) + k(f) + k(g) \geq 6 = 3n_0$$

as $k(e), k(g) \geq 1$ and $k(f) \geq 2$. Note that $\nu_2 = \frac{5}{6}n$ or $n = 3n_0$ if $k(e) = k(g) = 1$ and $k(f) = 2$.

Now, by induction, assume that for every graph G' obtained from a cubic pseudo-graph G_0' ($n_0' < n_0$) by $k'(e')$-subdividing each edge e' of G_0', we have

$$\nu_2' \geq \frac{5}{6}n' \quad \text{and} \quad n' \geq 3n_0',$$

and consider the cubic pseudo-graph G_0 ($n_0 \geq 4$) and its corresponding graph G.

Let e be a loop of G_0, and consider a cubic pseudo-graph G_0', obtained from G_0, by cutting the loop e ((a) of Fig. 1). Note that G_0' is well-defined, since $n_0 \geq 4$. As $n_0' < n_0$, due to induction hypothesis, we have

$$\nu_2' \geq \frac{5}{6}n' \quad \text{and} \quad n' \geq 3n_0', \tag{12}$$

where G' is obtained from G_0' by $k'(d')$-subdividing each edge d' of G_0', and the mapping k' is defined according to (10). On the other hand, due to (c1), (c2) and (c4), we have

$$\begin{aligned}&n_0 = n_0' + 2;\\ &n = n' + k(f) + k(e) + 4,\\ &\nu_2 \geq \nu_2' + k(f) + k(e) + 3.\end{aligned}$$

Since $k(f) \geq 2$, $k(e) \geq 1$ we have

$$\begin{aligned}&\frac{k(f) + k(e) + 3}{k(f) + k(e) + 4} \geq \frac{6}{7} > \frac{5}{6}, \quad \text{and}\\ &\frac{k(f) + k(e) + 4}{2} \geq \frac{7}{2} > 3\end{aligned}$$

and therefore due to (12) and Proposition 4, we get:

$$\begin{aligned}&\frac{\nu_2}{n} \geq \frac{\nu_2' + k(f) + k(e) + 3}{n' + k(f) + k(e) + 4} \geq \frac{5}{6}, \quad \text{and}\\ &\frac{n}{n_0} = \frac{n' + k(f) + k(e) + 4}{n_0' + 2} \geq 3.\end{aligned}$$

(e) We will prove (e1) by induction on n_0. Note that if $n_0 = 2$, then G_0 is the pseudo-graph from Fig. 2, thus

$$n = k(e) + k(f) + k(g) + 2 = \frac{k(e) + k(f) + k(g) + 2}{2} \cdot n_0$$

and due to (b)

$$\frac{\nu_2}{n} = \frac{k(e) + k(f) + k(g) + 1}{k(e) + k(f) + k(g) + 2}.$$

Now if G_0 satisfies (e1), then taking into account that $k(g) \geq 1$, $k(e) \geq 1$, $\max\{k(e), k(g)\} \geq 2$ and $k(f) \geq 2$, we get $k(e) + k(f) + k(g) \geq 5$, and therefore

$$\frac{\nu_2}{n} \geq \frac{6}{7} \quad \text{and} \quad n \geq \frac{7}{2} n_0.$$

Now, by induction, assume that for every graph G', obtained from a cubic pseudo-graph G_0' ($n_0' < n_0$), by $k'(e')$-subdividing each edge e' of G_0', we have

$$\nu_2' \geq \frac{6}{7} n' \quad \text{and} \quad n' \geq \frac{7}{2} n_0',$$

provided that G_0' satisfies (e1), and consider the cubic pseudo-graph G_0 ($n_0 \geq 4$) and its corresponding graph G. We need to consider two cases:

Case 1: G_0 contains at least two loops.

Let e_0 be a loop of G_0 that differs from e. Consider the cubic pseudo-graph G_0', obtained from G_0, by cutting the loop e_0 ((a) of Fig. 1), and the graph G', obtained from a cubic pseudo-graph G_0', by $k'(e')$-subdividing each edge e' of G_0', where the mapping k' is defined according to (10).

Since $n_0' < n_0$ and $e \in E_0'$, due to induction hypothesis, we have

$$\nu_2' \geq \frac{6}{7} n' \quad \text{and} \quad n' \geq \frac{7}{2} n_0'$$

(c1), (c2) and (c4) imply that

$$\begin{aligned} &n_0 = n_0' + 2; \\ &n = n' + k(f) + k(e_0) + 4, \\ &\nu_2 \geq \nu_2 + k(f) + k(e_0) + 3. \end{aligned}$$

Since $k(f) \geq 2$, $k(e_0) \geq 1$ we have

$$\begin{aligned} &\frac{k(f) + k(e_0) + 3}{k(f) + k(e_0) + 4} \geq \frac{6}{7}, \quad \text{and} \\ &\frac{k(f) + k(e_0) + 4}{2} \geq \frac{7}{2} \end{aligned}$$

and therefore due to Proposition 4, we get:

$$\begin{aligned} &\frac{\nu_2}{n} \geq \frac{\nu_2' + k(f) + k(e_0) + 3}{n' + k(f) + k(e_0) + 4} \geq \frac{6}{7}, \quad \text{and} \\ &\frac{n}{n_0} = \frac{n' + k(f) + k(e_0) + 4}{n_0' + 2} \geq \frac{7}{2}. \end{aligned}$$

Case 2: G_0 contains exactly one loop.

Let e – the only loop of G_0 – be adjacent to the edge d. Let u_0 be the vertex of G_0 that is incident to d and e, and let $d = (u_0, v_0)$. Let h and h' ($h \neq h'$) be two edges that differ from d and are incident to v_0. Finally, let u and v be the endpoints of h and h' that are not incident to d, respectively.

Subcase 2.1: $u \neq v$.

Consider a cubic pseudo-graph G_0' obtained from G_0 by cutting the loop e and the graph G' obtained from a cubic pseudo-graph G_0' by $k'(e')$-subdividing each edge e' of G_0', where the mapping k' is defined according to (10). As G_0' does not contain a loop, due to (a1) and (a2), we have

$$\nu_2' \geq \frac{7}{8} n' \quad \text{and} \quad n' \geq 4 n_0'. \tag{13}$$

(c1), (c2) and (c4) imply that

$$\begin{aligned} &n_0 = n_0' + 2; \\ &n = n' + k(d) + k(e) + 4, \\ &\nu_2 \geq \nu_2' + k(d) + k(e) + 3. \end{aligned}$$

Since $k(e) \geq 2$, $k(d) \geq 2$ we have

$$k(e) + k(d) \geq 4,$$

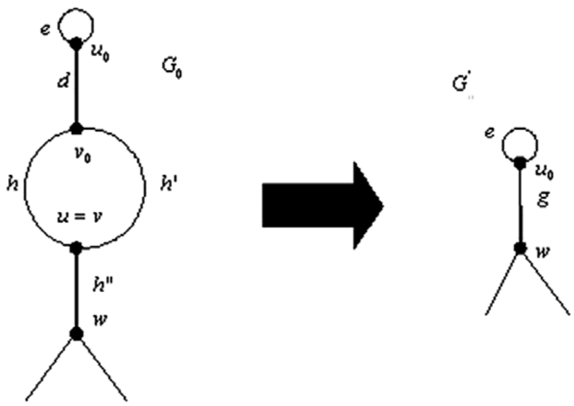

Fig. 3. Reducing G_0 to G_0'.

thus

$$\frac{k(d)+k(e)+3}{k(d)+k(e)+4} \geq \frac{7}{8} > \frac{6}{7}, \quad \text{and}$$
$$\frac{k(d)+k(e)+4}{2} \geq 4 > \frac{7}{2}.$$

Due to (13) and Proposition 4, we get:

$$\frac{\nu_2}{n} \geq \frac{\nu_2'+k(d)+k(e)+3}{n'+k(d)+k(e)+4} \geq \frac{6}{7}, \quad \text{and}$$
$$\frac{n}{n_0} = \frac{n'+k(d)+k(e)+4}{n_0'+2} \geq \frac{7}{2}.$$

Subcase 2.2: $u = v$.
Let h'' be the edge which is incident to u and is different from h and h', and let $h'' = (u, w)$ (Fig. 3).
Define a cubic pseudo-graph G_0' as follows:

$$G_0' = (G_0 \setminus \{v_0, u\}) \cup \{g\}, \quad \text{where}$$
$$g = (u_0, w),$$

and consider the graph G' obtained from G_0' by $k'(e')$-subdividing each edge e' of G_0', where

$$k'(e') = \begin{cases} k(d)+k(h'')-2 & \text{if } e' = g, \\ k(e') & \text{otherwise.} \end{cases}$$

Note that $e \in E_0'$, $n_0' < n_0$ and $k'(e) = k(e) \geq 2$ thus, due to induction hypothesis, we have:

$$\nu_2' \geq \frac{6}{7}n' \quad \text{and} \quad n' \geq \frac{7}{2}n_0'. \tag{14}$$

It is not hard to see that

$$n_0 = n_0' + 2;$$
$$n = n' + k(h) + k(h') + 4,$$
$$\nu_2 \geq \nu_2' + k(h) + k(h') + 3.$$

As $k(h), k(h') \geq 2$, we have

$$\frac{k(h)+k(h')+3}{k(h)+k(h')+4} \geq \frac{7}{8} > \frac{6}{7}, \quad \text{and}$$
$$\frac{k(h)+k(h')+4}{2} \geq 4 > \frac{7}{2},$$

therefore due to (14) and Proposition 4, we get:

$$\frac{\nu_2}{n} \geq \frac{\nu_2'+k(h)+k(h')+3}{n'+k(h)+k(h')+4} \geq \frac{6}{7}, \quad \text{and}$$

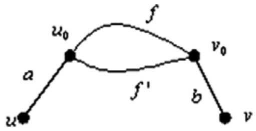

Fig. 4. The case of multiple edge.

$$\frac{n}{n_0} = \frac{n' + k(h) + k(h') + 4}{n'_0 + 2} \geq \frac{7}{2}.$$

The proof of (e1) is completed. Now, let us turn to the proof of (e2). Note that if G_0 does not contain a loop then (a1) and (a2) imply that

$$\nu_2 \geq \frac{7}{8}n > \frac{6}{7}n, \quad \text{and} \quad n \geq 4n_0 > \frac{7}{2}n_0,$$

thus, without loss of generality, we may assume that G_0 contains a loop. Our proof is by induction on n_0. Clearly, if $n_0 = 2$ then G_0 is the pseudo-graph from Fig. 2,

$$n = k(e) + k(f) + k(g) + 2 = \frac{k(e) + k(f) + k(g) + 2}{2} \cdot n_0$$

and due to (b)

$$\frac{\nu_2}{n} = \frac{k(e) + k(f) + k(g) + 1}{k(e) + k(f) + k(g) + 2}.$$

Now, if G_0 satisfies (e2) then $k(f) \geq 3$ and taking into account that $k(g) \geq 1$, $k(e) \geq 1$, we get $k(e) + k(f) + k(g) \geq 5$, therefore

$$\frac{\nu_2}{n} \geq \frac{6}{7} \quad \text{and} \quad n \geq \frac{7}{2}n_0.$$

Now, by induction, assume that for every graph G' obtained from a cubic pseudo-graph G'_0 ($n'_0 < n_0$) by $k'(e')$-subdividing each edge e' of G'_0, we have

$$\nu'_2 \geq \frac{6}{7}n' \quad \text{and} \quad n' \geq \frac{7}{2}n'_0$$

and consider the cubic pseudo-graph G_0 ($n_0 \geq 4$) and its corresponding graph G.

Case 1: There is an edge $f' = (u_0, v_0)$ such that f and f' form a cycle of the length two (Fig. 4).

Let $a, b, f, f', u_0, v_0, u, v$ be the edges and vertices as on Fig. 4. Consider a cubic pseudo-graph G'_0, defined as follows:

$$G'_0 = (G_0 \setminus \{u_0, v_0\}) \cup \{g\}, \quad \text{where}$$
$$g = (u, v),$$

and consider the graph G' obtained from G'_0 by $k'(e')$-subdividing each edge e' of G'_0, where

$$k'(e') = \begin{cases} k(f) & \text{if } e' = g, \\ k(e') & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} n_0 &= n'_0 + 2; \\ n &= n' + k(a) + k(b) + k(f') + 2, \\ \nu_2 &\geq \nu'_2 - (k(f) + 1) + k(a) + k(b) + k(f') + 2 + 1 + k(f) - 1 \\ &= \nu'_2 + k(a) + k(b) + k(f') + 1. \end{aligned}$$

Let us show that

$$\nu'_2 \geq \frac{6}{7}n' \quad \text{and} \quad n' \geq \frac{7}{2}n'_0.$$

First of all note that $n'_0 < n_0$ and $k'(g) = k(f) \geq 3$, therefore if g is not a loop of G'_0 ($u \neq v$) then the inequalities follow directly from the induction hypothesis. On the other hand, if g is a loop of G'_0 ($u = v$) then the same inequalities hold due to (e1).

Since

$$\frac{k(a)+k(b)+k(f')+1}{k(a)+k(b)+k(f')+2} \geq \frac{7}{8} > \frac{6}{7}, \quad \text{and}$$
$$\frac{k(a)+k(b)+k(f')+2}{2} \geq 4 > \frac{7}{2}.$$

Proposition 4 implies that

$$\frac{\nu_2}{n} \geq \frac{\nu_2'+k(a)+k(b)+k(f')+1}{n'+k(a)+k(b)+k(f')+2} \geq \frac{6}{7}, \quad \text{and}$$
$$\frac{n}{n_0} = \frac{n'+k(a)+k(b)+k(f')+2}{n_0'+2} \geq \frac{7}{2}.$$

Case 2: G_0 contains at least two loops and does not satisfy the condition of the Case 1.

As G_0 is connected and $n_0 \geq 4$, there is a loop e of G_0 such that e is not adjacent to f. Let d be the edge adjacent to the edge e. Let u_0 be the vertex of G_0 that is incident to d and e, and let $d = (u_0, v_0)$. Let h and h' be two edges that differ from d and are incident to v_0. Finally, let u and v be the endpoints of h and h' that are not incident to d, respectively.

Consider the cubic pseudo-graph G_0' obtained from G_0 by cutting the loop e and the graph G' obtained from a cubic pseudo-graph G_0' by $k'(e')$-subdividing each edge e' of G_0', where the mapping k' is defined according to (10). Note that $n_0' < n_0$.

Let us show that G_0' satisfies the condition of (e2). Clearly, if $f \in E_0'$ then we are done, thus we may assume that $f \notin E_0'$. Since $d \neq f$, we imply that $f \in \{h, h'\}$. As G_0 does not satisfy the condition of the Case 1, the edge $g \in E_0'$ is not a loop of G_0' and

$$k'(g) = k(h) + k(h') - 2 \geq 3.$$

Thus G_0' satisfies the condition of (e2), therefore, due to induction hypothesis, we get:

$$\nu_2' \geq \frac{6}{7}n' \quad \text{and} \quad n' \geq \frac{7}{2}n_0'.$$

(c1), (c2) and (c4) imply that

$$n_0 = n_0' + 2;$$
$$n = n' + k(d) + k(e) + 4,$$
$$\nu_2 \geq \nu_2' + k(d) + k(e) + 3.$$

Since $k(d) \geq 2$, $k(e) \geq 1$ we have

$$\frac{k(d)+k(e)+3}{k(d)+k(e)+4} \geq \frac{6}{7}, \quad \text{and}$$
$$\frac{k(d)+k(e)+4}{2} \geq \frac{7}{2}$$

therefore, due to Proposition 4, we get:

$$\frac{\nu_2}{n} \geq \frac{\nu_2'+k(d)+k(e)+3}{n'+k(d)+k(e)+4} \geq \frac{6}{7}, \quad \text{and}$$
$$\frac{n}{n_0} = \frac{n'+k(d)+k(e)+4}{n_0'+2} \geq \frac{7}{2}.$$

Case 3: G_0 contains exactly one loop e and does not satisfy the condition of the Case 1.

Let d be the edge adjacent to the edge e. Let u_0 be the vertex of G_0 that is incident to d and e, and let $d = (u_0, v_0)$. Let h and h' be two edges that differ from d and are incident to v_0. Finally, let u and v be the endpoints of h and h' that are not incident to d, respectively.

Subcase 3.1: $d = f$ and $u = v$.

Define a cubic pseudo-graph G_0' as follows (Fig. 3):

$$G_0' = (G_0 \setminus \{u, v_0\}) \cup \{g\}, \quad \text{where}$$
$$g = (u_0, w),$$

and consider the graph G' obtained from G_0' by $k'(e')$-subdividing each edge e' of G_0', where

$$k'(e') = \begin{cases} k(f) + k(h'') - 2 & \text{if } e' = g, \\ k(e') & \text{otherwise.} \end{cases}$$

Note that $n'_0 < n_0$ and $k'(g) = k(f) + k(h'') - 2 \geq 3$ thus, due to induction hypothesis, we have:

$$\nu'_2 \geq \frac{6}{7}n' \quad \text{and} \quad n' \geq \frac{7}{2}n'_0.$$

On the other hand, it is not hard to see that

$$n_0 = n'_0 + 2;$$
$$n = n' + k(h) + k(h') + 4,$$
$$\nu_2 \geq \nu'_2 + k(h) + k(h') + 3.$$

As $k(h), k(h') \geq 2$, we have

$$\frac{k(h) + k(h') + 3}{k(h) + k(h') + 4} \geq \frac{7}{8} > \frac{6}{7}, \quad \text{and}$$
$$\frac{k(h) + k(h') + 4}{2} \geq 4 > \frac{7}{2},$$

therefore, due to Proposition 4, we get:

$$\frac{\nu_2}{n} \geq \frac{\nu'_2 + k(h) + k(h') + 3}{n' + k(h) + k(h') + 4} \geq \frac{6}{7}, \quad \text{and}$$
$$\frac{n}{n_0} = \frac{n' + k(h) + k(h') + 4}{n'_0 + 2} \geq \frac{7}{2}.$$

Subcase 3.2: $d \neq f$ or $u \neq v$.

Consider the cubic pseudo-graph G'_0 obtained from G_0 by cutting the loop e and the graph G' obtained from a cubic pseudo-graph G'_0 by $k'(e')$-subdividing each edge e' of G'_0, where the mapping k' is defined according to (10). Note that $n'_0 < n_0$.

Let us show that G'_0 and its corresponding graph G' satisfy

$$\nu'_2 \geq \frac{6}{7}n' \quad \text{and} \quad n' \geq \frac{7}{2}n'_0. \tag{15}$$

Note that if $f \in E'_0$, then, since $n'_0 < n_0$ and $k'(f) = k(f) \geq 3$, (15) follows directly from the induction hypothesis. So, let us assume, that $f \notin E'_0$. If $d = f$ then G'_0 does not contain a loop as $u \neq v$. Thus (15) follows from (a1) and (a2). Thus, we may also assume that $d \neq f$. As $f \notin E'_0$, we deduce that $f \in \{h, h'\}$. As G_0 does not satisfy the condition of the Case 1, we have $u \neq v$ and G'_0 does not contain a loop. Thus (15) again follows from (a1) and (a2).

Now, (c1), (c2) and (c4) imply that

$$n_0 = n'_0 + 2;$$
$$n = n' + k(d) + k(e) + 4,$$
$$\nu_2 \geq \nu'_2 + k(d) + k(e) + 3.$$

Since $k(d) \geq 2$, $k(e) \geq 1$, we have

$$\frac{k(d) + k(e) + 3}{k(d) + k(e) + 4} \geq \frac{6}{7}, \quad \text{and}$$
$$\frac{k(d) + k(e) + 4}{2} \geq \frac{7}{2}$$

therefore, due to (15) and Proposition 4, we get:

$$\frac{\nu_2}{n} \geq \frac{\nu'_2 + k(d) + k(e) + 3}{n' + k(d) + k(e) + 4} \geq \frac{6}{7}, \quad \text{and}$$
$$\frac{n}{n_0} = \frac{n' + k(d) + k(e) + 4}{n'_0 + 2} \geq \frac{7}{2}.$$

(f) Note that if G_0 satisfies at least one of the conditions of (a), (e1), (e2), then, taking into account the inequality $2\nu_1 \geq \nu_2$, we get:

$$\nu_1 \geq \frac{\nu_2}{2} \geq \frac{1}{2} \cdot \frac{6}{7}n = \frac{3}{7}n,$$

thus, without loss of generality, we may assume that G_0 satisfies none of the conditions of (a), (e1), (e2), hence G_0 contains at least one loop, and for each loop e and for each edge f of G_0, that is not a loop, we have: $k(e) = 1$ and $k(f) = 2$. For these

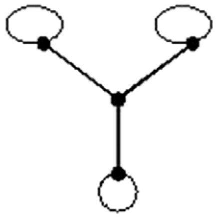

Fig. 5. The case $k = 1$.

cubic pseudo-graphs, we will prove the inequality (f) by induction on n_0. If $n_0 = 2$, then G_0 is the cubic pseudo-graph from the Fig. 2 and, as $k(e) = k(g) = 1$ and $k(f) = 2$, G contains a perfect matching, thus

$$\nu_1 = \frac{1}{2}n > \frac{3}{7}n.$$

Now, by induction, assume that for every graph G' obtained from a cubic pseudo-graph G_0' ($n_0' < n_0$) by $k'(e')$-subdividing each edge e' of G_0', we have

$$\nu_1' \geq \frac{3}{7}n',$$

and consider the cubic pseudo-graph G_0 ($n_0 \geq 4$) and its corresponding graph G.

Let e be a loop of G_0, and consider a cubic pseudo-graph G_0' obtained from G_0 by cutting the loop e and a graph G' obtained from G_0' by $k'(d')$-subdividing each edge d' of G_0', where the mapping k' is defined according to (10). As $n_0' < n_0$, due to induction hypothesis, we have

$$\nu_1' \geq \frac{3}{7}n'$$

(c2) and (c3) imply that

$$n = n' + 7,$$
$$\nu_1 \geq \nu_1' + 3.$$

Due to Proposition 4, we get:

$$\frac{\nu_1}{n} \geq \frac{\nu_1' + 3}{n' + 7} \geq \frac{3}{7}.$$

(g) Let G_0 be the connected cubic pseudo-graph corresponding to G and let $\bar{G}_0$ be the tree obtained from G_0 by removing its loops (see the definition of the class $\mathfrak{M}$). Assume k and k' to be the numbers of internal (non-pendant) and pendant vertices of $\bar{G}_0$. Clearly, $k + k' = \bar{n}_0 = n_0$. On the other hand,

$$\bar{m}_0 = m_0 - k' = \frac{3}{2}(k + k') - k'.$$

Since $\bar{m}_0 = \bar{n}_0 - 1$, we get

$$k + k' - 1 = \frac{3}{2}(k + k') - k'$$

or

$$k' = k + 2.$$

We prove the inequality by induction on k. Note that if $k = 0$ then G_0 is the cubic pseudo-graph from the Fig. 2, therefore

$$\frac{\nu_1}{n} = \frac{3}{6} = \frac{1}{2} > \frac{6}{13}.$$

On the other hand, if $k = 1$, then G_0 is the cubic pseudo-graph shown on the Fig. 5, thus

$$\frac{\nu_1}{n} = \frac{6}{13}.$$

Now, by induction, assume that for every graph $G' \in \mathfrak{M}$, we have $\nu_1' \geq \frac{6}{13}n'$, if the tree $\bar{G}_0'$ contains less than k internal vertices, and let us consider the graph $G \in \mathfrak{M}$ the corresponding tree $\bar{G}_0$ of which contains k ($k \geq 2$) internal vertices. We need to consider two cases:

Case 1: There is $U = \{u_1, \ldots, u_7\} \subseteq \bar{V}_0$ such that $d_{\bar{G}_0}(u_i) = 1,\ 1 \leq i \leq 4$ and the subtree of $\bar{G}_0$ induced by U is the tree shown on the Fig. 6.

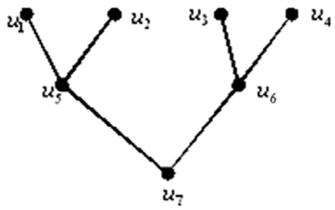

Fig. 6. The case of two branches.

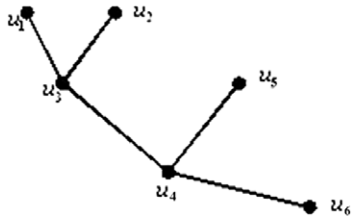

Fig. 7. The case of a branch and a leave.

Let $\bar{G}'_0$ be the tree $\bar{G}_0 \setminus \{u_1, \ldots, u_6\}$ and let G'_0 be the cubic pseudo-graph obtained from G_0 by removing the vertices $u_1, \ldots, u_6$ and adding a new loop incident to the vertex u_7. Note that $\bar{G}'_0$ contains less than k internal vertices, thus for the graph $G' \in \mathfrak{M}$ corresponding to $\bar{G}'_0$, we have

$$\frac{\nu'_1}{n'} \geq \frac{6}{13}. \tag{16}$$

On the other hand, since

$$n = n' - 1 + (6 + 16) = n' + 21,$$
$$\nu_1 \geq \nu'_1 - 1 + 11 = \nu'_1 + 10,$$

due to (16) and Proposition 4, we get:

$$\frac{\nu_1}{n} \geq \frac{\nu'_1 + 10}{n' + 21} \geq \frac{6}{13}$$

since

$$\frac{10}{21} > \frac{6}{13}.$$

Case 2: There is $U = \{u_1, \ldots, u_6\} \subseteq \bar{V}_0$ such that $d_{\bar{G}_0}(u_1) = d_{\bar{G}_0}(u_2) = d_{\bar{G}_0}(u_5) = 1$ and the subtree of $\bar{G}_0$ induced by U is the tree shown on the Fig. 7.

Let $\bar{G}'_0$ be the tree $(\bar{G}_0 \setminus \{u_1, \ldots, u_4\}) \cup \{(u_5, u_6)\}$ and let G'_0 be the cubic pseudo-graph obtained from G_0 by removing the vertices u_1, u_2, u_3 and u_4 and adding the edge (u_5, u_6). Note that $\bar{G}'_0$ contains less than k internal vertices, thus for the graph $G' \in \mathfrak{M}$ corresponding to $\bar{G}'_0$, we have

$$\frac{\nu'_1}{n'} \geq \frac{6}{13}. \tag{17}$$

On the other hand, since

$$n = n' - 2 + 14 = n' + 12,$$
$$\nu_1 \geq \nu'_1 - 1 + 8 = \nu'_1 + 7,$$

due to (16) and Proposition 4, we get:

$$\frac{\nu_1}{n} \geq \frac{\nu'_1 + 7}{n' + 12} \geq \frac{6}{13},$$

since

$$\frac{7}{12} > \frac{6}{13}.$$

To complete the proof of the inequality, let us note that, since the tree $\bar{G}_0$ contains k, $(k \geq 2)$ internal vertices, $\bar{G}_0$ satisfies at least one of the conditions of Case 1 and Case 2.

(h) (c1) and (c2) imply that

$$n_0 = n_0' + 2;$$
$$n = n' + k(f) + k(e) + 4.$$

Since $k(f) \geq 2$, $k(e) \geq 1$ we have

$$\frac{k(f) + k(e) + 4}{2} \geq \frac{7}{2},$$

thus, due to Proposition 4, we get:

$$\frac{n}{n_0} = \frac{n' + k(f) + k(e) + 4}{n_0' + 2} \geq \frac{7}{2}.$$

(i) Note that as $n < \frac{7}{2} n_0$ due to (e1) and (e2), for every edge e of G_0 we have

$$k(e) = \begin{cases} 1, & \text{if } e \text{ is a loop,} \\ 2, & \text{otherwise.} \end{cases}$$

Let us show that $G \in \mathfrak{M}$. Consider a maximal (with respect to the operation of cutting loops) sequence of cubic pseudo-graphs $G_0^{(0)}, G_0^{(1)}, \ldots, G_0^{(n)}$, where $G_0^{(0)} = G_0$, and $G_0^{(i+1)}$ is obtained from $G_0^{(i)}$ by cutting a loop e_i of $G_0^{(i)}$, $i = 0, \ldots, n-1$. Note that Proposition 2 implies that for $i = 1, \ldots, n$ the graph $G_0^{(i)}$ is connected.

Consider the sequence of graphs $G^{(0)}, G^{(1)}, \ldots, G^{(n)}$, where $G^{(0)} = G$, and for $i = 1, \ldots, n$ the graph G_i is obtained from $G_0^{(i)}$ by $k_i(d_i)$-subdividing each edge d_i of $G_0^{(i)}$, where the mapping k_i is defined from k_{i-1} according to (10) and $k_0 = k$. As the sequence $G_0^{(0)}, G_0^{(1)}, \ldots, G_0^{(n)}$ is maximal, the operation of cutting the loops is not applicable to $G_0^{(n)}$, thus due to Remark 1, $G_0^{(n)}$ is either the trivial cubic pseudo-graph from the Fig. 2 or a connected graph (i.e. a connected pseudo-graph without loops). On the other hand, (h) implies that for $i = 1, \ldots, n$, we have

$$n^{(i)} < \frac{7}{2} n_0^{(i)} \tag{18}$$

thus, taking into account (a2), we deduce that $G_0^{(n)}$ is the trivial cubic pseudo-graph from the Fig. 2.

Note that for the proof of $G \in \mathfrak{M}$, it suffices to show that if we remove all loops of G_0 then we will get a tree, which is equivalent to proving that G_0 does not contain a cycle. Suppose that G_0 contains a cycle. As $G_0^{(n)}$, which is the pseudo-graph from the Fig. 2, does not contain a cycle, we imply that there is j, $1 \leq j \leq n-1$ such that $G_0^{(j)}$ contains a cycle and $G_0^{(j+1)}$ does not. Proposition 3 implies that the loop e_j of $G_0^{(j)}$, whose cut led to the cubic pseudo-graph $G_0^{(j+1)}$ is adjacent to an edge f_j which, in its turn, is adjacent to two edges h_j and h_j' that form the only cycle of $G_0^{(j)}$.

As the edges h_j and h_j' form a cycle of $G_0^{(j)}$, the cut of the loop e_j leads to a loop g_{j+1} of $G_0^{(j+1)}$ (see the definition of the operation of the cut of loops). Due to (10), we have

$$k_{j+1}(g_{j+1}) = k_j(h_j) + k_j(h_j') - 2 = 2$$

thus, due to (e1), we have

$$n^{(j+1)} \geq \frac{7}{2} n_0^{(j+1)}$$

contradicting (18).The proof of the lemma is completed. □

7. The main results

We are ready to prove the first result of the paper. The basic idea of the proof of this theorem can be roughly described as follows: proving a lower bound for the main parameters of a cubic graph G is just proving a bound for the graph $G \setminus F$ obtained by removing a maximum matching F of G. Next, according to Lemma 2, there is a maximum matching of a cubic graph such that its removal leaves a graph, in which each degree is either two or three. Moreover, the vertices of degree three are not placed very closed. This allows us to consider this graph as a subdivision of a cubic pseudo-graph, in which each edge is subdivided sufficiently many times. The word "sufficiently" here should be understood as big enough to allow us to apply the main results of the Lemma 6. Next, by considering the connected components of $G \setminus F$, we divide them into two or three groups. For each of this groups, thanks to Lemma 6, we find a bound for our parameters. Then, due to Proposition 5, we not only estimate the total contribution of the connected components to the main parameters, but also keep this estimations big enough, which allows us to get the main results of the theorem.

Theorem 4. *Let G be a cubic graph. Then:*

$$\nu_1 \geq \frac{2}{5}n, \qquad \nu_2 \geq \frac{4}{5}n, \qquad \nu_3 \geq \frac{7}{6}n.$$

Proof. In [16] it is shown that every odd regular graph G contains a matching of size at least $\left\lceil \frac{(r^2-r-1)n-(r-1)}{r(3r-5)} \right\rceil$, where r is the degree of vertices of G. Particularly, for a cubic graph G we have:

$$\nu_1 \geq \left\lceil \frac{5n-2}{12} \right\rceil \geq \frac{2}{5}n.$$

Now, let us show that the other two inequalities are also true. Let F be a maximum matching of G such that the unsaturated vertices (with respect to F) do not have a common neighbour (see Lemma 2). Let ε be a rational number such that $\varepsilon \in [0, \frac{1}{10}]$ and

$$\nu_1 = |F| = \left(\frac{2}{5} + \varepsilon\right) n.$$

Note that to complete the proof, it suffices to show that

$$\nu_1(G \setminus F) \geq \left(\frac{2}{5} - \varepsilon\right) n, \qquad \nu_2(G \setminus F) \geq \left(\frac{23}{30} - \varepsilon\right) n.$$

Consider the graph $G \setminus F$. Clearly,

$$2 = \delta(G \setminus F) \leq \Delta(G \setminus F) \leq 3.$$

Let x and y be the numbers of vertices of $G \setminus F$ with degree two and three, respectively. Clearly,

$$\begin{cases} x + y = |V(G \setminus F)| = n, \\ 2x + 3y = 2m - 2\,|F| = 3n - \left(\frac{4}{5} + 2\varepsilon\right) n = \left(\frac{11}{5} - 2\varepsilon\right) n, \end{cases}$$

which implies that

$$x = \left(\frac{4}{5} + 2\varepsilon\right) n, \qquad y = \left(\frac{1}{5} - 2\varepsilon\right) n.$$

Let $G_1, \dots, G_r$ be the connected components of $G \setminus F$. For a vertex $v_i \in V_i$, $1 \leq i \leq r$ define:

$$\nu_1(v_i) = \frac{\nu_{1i}}{n_i}, \qquad \nu_2(v_i) = \frac{\nu_{2i}}{n_i}.$$

Note that

$$\begin{aligned} \frac{\nu_1(G \setminus F)}{|V(G \setminus F)|} = \frac{\nu_1(G \setminus F)}{n} &= \frac{\nu_{1,1} + \cdots + \nu_{1,r}}{n_1 + \cdots + n_r} \\ &= \frac{n_1 \cdot \frac{\nu_{1,1}}{n_1} + \cdots + n_r \cdot \frac{\nu_{1,r}}{n_r}}{n_1 + \cdots + n_r} \\ &= \frac{n_1 \cdot \nu_1(v_1) + \cdots + n_r \cdot \nu_1(v_r)}{n_1 + \cdots + n_r}, \end{aligned} \tag{19}$$

and similarly

$$\frac{\nu_2(G \setminus F)}{|V(G \setminus F)|} = \frac{n_1 \cdot \nu_2(v_1) + \cdots + n_r \cdot \nu_2(v_r)}{n_1 + \cdots + n_r} \tag{20}$$

where $v_1, \dots, v_r$ are vertices of $G \setminus F$ with $v_i \in V(G_i)$, $1 \leq i \leq r$.

By the choice of F, we have that for $i = 1, \dots, r$ G_i is

(a) either a cycle,
(b) or a connected graph, with $\delta_i = 2$, $\Delta_i = 3$ which does not contain two vertices of degree three that are adjacent or share a neighbour.

Note that if G_i is of type (b), then there is a cubic pseudo-graph G_i^0 such that G_i can be obtained from G_i^0 by $k(e)$-subdividing each edge e of G_i^0 (Proposition 1). Of course, if e is not a loop then $k(e) \geq 2$.

Let a, b, c be the numbers of vertices of $G \setminus F$ that lie on its connected components $G_1, \dots, G_r$, which are cycles, graphs of type (b) that are from the class $\mathfrak{M}$, graphs of type (b) which are not from the class $\mathfrak{M}$, respectively.

It is clear that if v_a is a vertex of $G \setminus F$ lying on a cycle of length l then

$$\nu_1(v_a) = \frac{\left[\frac{l}{2}\right]}{l} \geq \frac{1}{3}.$$

If v_b is a vertex of $G \setminus F$ lying on a connected component G_b of $G \setminus F$ which is from the class $\mathfrak{M}$, then (g) of Lemma 6 implies that

$$\nu_1(v_b) = \frac{\nu_{1b}}{n_b} \geq \frac{6}{13}.$$

If v_c is a vertex of $G \setminus F$ lying on a connected component G_c of $G \setminus F$ which is of type (b) and does not belong to the class $\mathfrak{M}$, then (f) of Lemma 6 implies that

$$\nu_1(v_c) = \frac{\nu_{1c}}{n_c} \geq \frac{3}{7}.$$

Let k_b and k_c be the number of vertices of $G \setminus F$ with degree three that lie on connected components $G_1, \ldots, G_r$, which are graphs from the class $\mathfrak{M}$ or are graphs of type (b), which are not from the class $\mathfrak{M}$, respectively. Clearly,

$$k_b + k_c = y = \left(\frac{1}{5} - 2\varepsilon\right) n. \tag{21}$$

(d2) of Lemma 6 implies that

$$b \geq 3k_b.$$

(i) of Lemma 6 implies that

$$c \geq \frac{7}{2}k_c.$$

Thus, due to (19)

$$\frac{\nu_1(G \setminus F)}{|V(G \setminus F)|} \geq \frac{\frac{1}{3}a + \frac{6}{13}b + \frac{3}{7}c}{n}.$$

As $a + b + c = n$ we get: $a \leq n - 3k_b - \frac{7}{2}k_c$. Since $\frac{1}{3} < \frac{3}{7} < \frac{6}{13}$, due to Proposition 5, we have:

$$\frac{1}{3}a + \frac{6}{13}b + \frac{3}{7}c \geq \frac{1}{3}\left(n - 3k_b - \frac{7}{2}k_c\right) + \frac{6}{13} \cdot 3k_b + \frac{3}{7} \cdot \frac{7}{2}k_c$$

and therefore

$$\frac{\nu_1(G \setminus F)}{|V(G \setminus F)|} \geq \frac{\frac{1}{3}\left(n - 3k_b - \frac{7}{2}k_c\right) + \frac{6}{13} \cdot 3k_b + \frac{3}{7} \cdot \frac{7}{2}k_c}{n}$$
$$= \frac{\frac{1}{3}n + \frac{5}{13}k_b + \frac{1}{3}k_c}{n} = \frac{1}{3} + \frac{1}{3}\frac{k_b + k_c}{n} + \frac{2}{39}\frac{k_b}{n}$$

(21) implies that

$$\frac{\nu_1(G \setminus F)}{n} \geq \frac{1}{3} + \frac{1}{3}\left(\frac{1}{5} - 2\varepsilon\right) + \frac{2}{39}\frac{k_b}{n} = \frac{2}{5} - \frac{2}{3}\varepsilon + \frac{2}{39}\frac{k_b}{n}$$
$$= \left(\frac{2}{5} - \varepsilon\right) + \frac{\varepsilon}{3} + \frac{2}{39}\frac{k_b}{n} \geq \frac{2}{5} - \varepsilon$$

which is equivalent to

$$\nu_1(G \setminus F) \geq \left(\frac{2}{5} - \varepsilon\right)|V(G \setminus F)| = \left(\frac{2}{5} - \varepsilon\right) n.$$

Note that if $\nu_2 = \frac{4}{5}n$, then $\varepsilon = 0$, $k_b = 0$, which means that $\nu_1 = \frac{2}{5}n$ and among the components $G_1, \ldots, G_r$ there are no representatives of the class $\mathfrak{M}$.

Now, let us turn to the proof of the inequality $\nu_2(G \setminus F) \geq \left(\frac{23}{30} - \varepsilon\right) n$.

Let A, B be the numbers of vertices of $G \setminus F$ that lie on its connected components $G_1, \ldots, G_r$, which are cycles and graphs of type (b), respectively. It is clear that if v_A is a vertex of $G \setminus F$ lying on a cycle of the length l then

$$\nu_2(v_A) = \frac{2\left[\frac{l}{2}\right]}{l} \geq \frac{2}{3}.$$

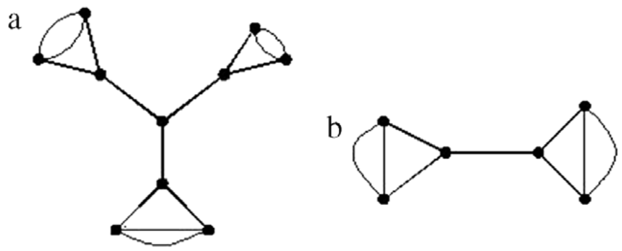

Fig. 8. Examples attaining the bounds of the Theorem 4.

If v_B is a vertex of $G \setminus F$ lying on a connected component G_B of $G \setminus F$ which is of type (b), then (d1) of Lemma 6 implies that

$$\nu_2(v_B) = \frac{\nu_{2B}}{n_B} \geq \frac{5}{6}.$$

As the number of vertices of $G \setminus F$ which are of degree three is $y = \left(\frac{1}{5} - 2\varepsilon\right) n$, (d2) of Lemma 6 implies that

$$B \geq 3y = \left(\frac{3}{5} - 6\varepsilon\right) n. \tag{22}$$

Thus, due to (20),

$$\frac{\nu_2(G \setminus F)}{|V(G \setminus F)|} \geq \frac{\frac{2}{3}A + \frac{5}{6}B}{|V(G \setminus F)|}.$$

As $A + B = n$, (22) implies that

$$A \leq n - 3y = n - \left(\frac{3}{5} - 6\varepsilon\right) n = \left(\frac{2}{5} + 6\varepsilon\right) n.$$

Since $\frac{2}{3} < \frac{5}{6}$, due to Proposition 5, we get

$$\frac{2}{3}A + \frac{5}{6}B \geq \frac{2}{3}\left(\frac{2}{5} + 6\varepsilon\right) n + \frac{5}{6}\left(\frac{3}{5} - 6\varepsilon\right) n$$

and therefore

$$\frac{\nu_2(G \setminus F)}{|V(G \setminus F)|} \geq \frac{\left(\frac{23}{30} - \varepsilon\right) n}{|V(G \setminus F)|} = \left(\frac{23}{30} - \varepsilon\right),$$

which is equivalent to

$$\nu_2(G \setminus F) \geq \left(\frac{23}{30} - \varepsilon\right) |V(G \setminus F)| = \left(\frac{23}{30} - \varepsilon\right) n.$$

The proof of the theorem is completed. □

Remark 3. There are graphs attaining bounds of the Theorem 4. The graph from Fig. 8(a) attains the first two bounds and the graph from Fig. 8(b) the last bound.

Recently, we managed to prove:

Theorem 5. *For every cubic graph G*

$$\nu_2 + \nu_3 \geq 2n.$$

Note that this implies that there is no graph attaining all the bounds of Theorem 4 at the same time.

The methodology developed above allows us to prove the second result of the paper, which is an inequality among our main parameters. To prove it, again we reduce the inequality to another one considered in the class of graphs, that are obtained from a cubic graph by removing a matching of G. Note that this time matching need not to be maximum, nevertheless, its removal keeps the vertices of degree three "far enough". Next, by considering any of connected components of this graph, we look at it as a subdivision of a cubic pseudo-graph. This allows us to apply the results from the section on system of cycles and paths, and find a suitable system, which not only captures the essence of the inequality that we were trying to prove, but also is very simple in its structure, and this allows us to complete the proof.

Theorem 6. *For every cubic graph G the following inequality holds:*

$$\nu_2 \leq \frac{n + 2\nu_3}{4}.$$

Proof. Let (H, H') be a pair of edge-disjoint matchings of G with $|H| + |H'| = \nu_2$. Without loss of generality we may assume that H is maximal (not necessarily maximum). Let $G_1, \ldots, G_k$ be the connected components of $G \backslash H$, $l_i = l(G_i)$ be the number of vertices of G_i having degree three, $1 \leq i \leq k$, and let l be the number of vertices of $G \setminus H$ having degree three. Note that

$$l = l_1 + \cdots + l_k = n - 2|H|.$$

Let us show that for each i, $1 \leq i \leq k$, the following inequality is true:

$$\nu_{2i} \geq 2\nu_{1i} - \frac{l_i}{2}. \tag{23}$$

Note that, if G_i is a cycle, then $l_i = 0$ and $\nu_{2i} = 2\nu_{1i}$, thus (23) is true for the cycles. Now, let us assume G_i to contain a vertex of degree three. As H is a maximal matching, no two vertices of degree three are adjacent in G_i. Proposition 1 implies that there is a cubic pseudo-graph G_i^0 such that G_i can be obtained from G_i^0 by $k(e)$-subdividing each edge e of G_i^0 where $k(e) \geq 1$. Let G_i' be the graph obtained from G_i^0 by 1-subdividing each edge e of G_i^0. Note that G_i' contains n_i^0 vertices of degree three, $\frac{3n_i^0}{2}$ vertices of degree two and no two vertices of the same degree are adjacent in G_i'. Due to Lemma 3, there is a system $\mathfrak{F}_i'$ of even cycles and paths of G_i' satisfying the conditions (1.2), (1.3) of the Lemma 3 and containing $\frac{n_i^0}{2}$ paths (see (1.1) of the Lemma 3). (2) of Lemma 3 implies that $\mathfrak{F}_i'$ includes a maximum matching of G_i'.

Now, note that G_i can be obtained from G_i' by a sequence of 1-subdivisions. Lemma 4 implies that there is a system $\mathfrak{F}_i$ of paths and even cycles of G_i satisfying the conditions (1)–(5) of the Lemma 3 and containing exactly $\frac{n_i^0}{2}$ paths!

Let x be the number of paths from $\mathfrak{F}_i$ containing an odd number of edges. Note that since $x \leq \frac{n_i^0}{2}$, we have:

$$\begin{aligned}\nu_{2i} &\geq \sum_{F \in \mathfrak{F}_i} |E(F)| = 2 \sum_{F \in \mathfrak{F}_i} \nu_1(F) - x = 2\nu_{1i} - x \\ &\geq 2\nu_{1i} - \frac{n_i^0}{2} = 2\nu_{1i} - \frac{l_i}{2}.\end{aligned}$$

Summing up the inequalities (23) from 1 to k we get:

$$\nu_2(G \setminus H) = \sum_{i=1}^{k} \nu_{2i} \geq 2 \sum_{i=1}^{k} \nu_{1i} - \frac{\sum_{i=1}^{k} l_i}{2} = 2\nu_1(G \setminus H) - \frac{l}{2}.$$

Thus

$$\nu_3 \geq |H| + \nu_2(G \setminus H) \geq |H| + 2\nu_1(G \setminus H) - \frac{l}{2} = |H| + 2\nu_1(G \setminus H) - \frac{n}{2} + |H|.$$

Taking into account that

$$|H| + |H'| = |H| + \nu_1(G \setminus H) = \nu_2$$

we get:

$$\nu_3 \geq 2\nu_2 - \frac{n}{2}$$

or

$$\nu_2 \leq \frac{n + 2\nu_3}{4}.$$

The proof of the Theorem 6 is completed. □

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